EXAM

Exam #1

Math 1351-007, Fall 2002

October 2, 2002

ANSWERS

90 pts.

Problem 1. Figure 1 shows the graph of a function f. In each part, find the right-hand limit, left-hand limit and two-sided limit as x approaches a; find the value $f(a)$; and determine if the function is continuous at a. Justify your answers.

A. $a = -2$.

Answer:

We have

$$\lim_{x \to -2^-} f(x) = 4, \qquad \lim_{x \to -2^+} f(x) = 2$$

and so

$$\lim_{x \to -2} f(x) \quad \text{Does not Exist.}$$

The value of the function at -2 is $f(-2) = 4$. The function is not continuous at $x = -2$, because the limit does not exist.

B. $a = 0$.

Answer:

We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 4 = f(0),$$

so the function is continuous at $x = 0$.

C. $a = 2$.

Answer:

We have

$$\lim_{x \to 2^-} f(x) = 6 = \lim_{x \to 2^+} f(x)$$

and so

$$\lim_{x \to 2} f(x) = 6.$$

The value of the function is $f(2) = 4$. The function is not continuous at $x = 2$, since

$$\lim_{x \to 2} f(x) \neq f(2).$$

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50 pts.

Problem 2. Consider the function

$$f(x) = \begin{cases} x + 1, & x < 0 \\ 1, & x = 0 \\ 2x + 2, & 0 < x < 3 \\ x^2 - 5, & 3 \leq x. \end{cases}$$

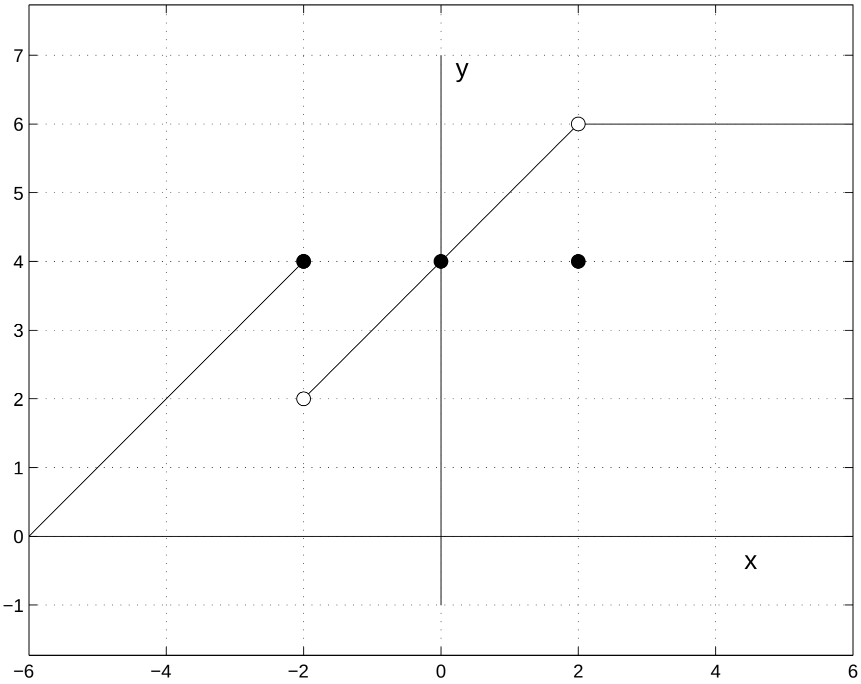

Figure 1: The function for Problem 1

Find the suspicious points and determine if f is continuous at each suspicious point. Explain your answers.

Answer:
The suspicious points are $x = 0$ and $x = 3$, since the formula changes at these points. For $x = 0$ we have

$$\begin{gathered}\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+1) = 0+1 = 1\\ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(2x+2) = 2(0)+2 = 2\\ \lim_{x\to 0} f(x) \text{ does not exist}\\ f(0) = 1.\end{gathered}$$

The function is not continuous at $x = 0$, because $\lim_{x\to 0} f(x)$ does not exist.

For $x = 3$, we have

$$\begin{gathered}\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(2x+2) = 2(3)+2 = 8\\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2-5) = 3^2-5 = 4\\ \lim_{x\to 3} f(x) \text{ does not exist}\\ f(3) = 3^2-5 = 4.\end{gathered}$$

The function is not continuous at $x = 3$, because the limit does not exist.

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80 pts.

Problem 3. In each part, find the limit (if it exists).

A.

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}, \qquad \text{(note correction)}$$

Answer:

We have

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2}(x+3) = 2+3 = 5.$$

B.

$$\lim_{x\to 0}\frac{x}{\sin(2x)}$$

Answer:

We calculate as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(2x)} &= \lim_{x\to 0}\frac{2x}{2\sin(2x)}\\ &= \frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin(2x)}\\ &= \frac{1}{2}\lim_{x\to 0}\frac{1}{\frac{\sin(2x)}{2x}}\\ &= \frac{1}{2}\frac{1}{1}\\ &= \frac{1}{2},\end{aligned}$$

using the limit

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1 \tag{$*$}$$

from class.

C.

$$\lim_{x\to 0}\frac{\tan(3x)}{\tan(x)}$$

Answer:

Putting in lots of detail, we can calculate as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(3x)}{\tan(x)} &= \lim_{x\to 0}\frac{\frac{\sin(3x)}{\cos(3x)}}{\frac{\sin(x)}{\cos(x)}}\\ &= \lim_{x\to 0}\frac{\cos(x)}{\sin(x)}\frac{\sin(3x)}{\cos(3x)}\\ &= \lim_{x\to 0}\frac{\cos(x)}{\cos(3x)}\frac{\sin(3x)}{\sin(x)}\\ &= \lim_{x\to 0}\frac{\cos(x)}{\cos(3x)}\frac{\sin(3x)}{\sin(x)}\frac{\frac{1}{x}}{\frac{1}{x}}\\ &= \lim_{x\to 0}\frac{\cos(x)}{\cos(3x)}\frac{\frac{\sin(3x)}{x}}{\frac{\sin(x)}{x}}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\cos(x)}{\cos(3x)} \frac{3\dfrac{\sin(3x)}{3x}}{\dfrac{\sin(x)}{x}} \\
&= \frac{1}{1}\frac{3(1)}{1} \\
&= 3,
\end{aligned}
$$

Using the limit $(*)$ and the fact that cos is a continuous function with $\cos(0) = 1$.

D.

$$\lim_{x\to 1} \frac{\sqrt{2x+2}-2}{x-1}$$

Answer:

We make use of the identity $(a-b)(a+b) = a^2 - b^2$ and calculate as follows:

$$
\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{2x+2}-2}{x-1} &= \lim_{x\to 1} \frac{\sqrt{2x+2}-2}{x-1}\ \frac{\sqrt{2x+2}+2}{\sqrt{2x+2}+2} \\
&= \lim_{x\to 1} \frac{(\sqrt{2x+2})^2 - 2^2}{(x-1)(\sqrt{2x+2}+2)} \\
&= \lim_{x\to 1} \frac{2x+2-4}{(x-1)(\sqrt{2x+2}+2)} \\
&= \lim_{x\to 1} \frac{2x-2}{(x-1)(\sqrt{2x+2}+2)} \\
&= \lim_{x\to 1} \frac{2(x-1)}{(x-1)(\sqrt{2x+2}+2)} \\
&= \lim_{x\to 1} \frac{2}{\sqrt{2x+2}+2} \\
&= \frac{2}{\sqrt{2(1)+2}+2} \\
&= \frac{2}{4} \\
&= \frac{1}{2}
\end{aligned}
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40 pts.

Problem 4. Solve the following equation:

$$\ln(x) + \ln(2x+1) = 0.$$

Answer:
We solve the equation by the following sequence of steps:

$$\begin{aligned}
(*) \qquad \ln(x) + \ln(2x+1) &= 0 \\
\ln(x(2x+1)) &= 0 && \text{product rule for logs} \\
e^{\ln(x(2x+1))} &= e^0 && \text{take exponential of both sides} \\
x(2x+1) &= 1 && \text{inverse relations, } e^0 = 1 \\
2x^2 + x &= 1 \\
2x^2 + x - 1 &= 0 \\
(2x-1)(x+1) &= 0
\end{aligned}$$

The solutions of the last equation are $x = -1$ and $x = 1/2$. However, if we plug $x = -1$ into the original equation $(*)$, the first term becomes $\ln(-1)$, which is undefined. Thus, $x = -1$ is *not* a solution of the original equation $(*)$. If we plug $x = 1/2$ into $(*)$, both of the expressions inside the logs are positive, and so defined. We conclude that the only solution of $(*)$ is $x = 1/2$.

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40 pts.

Problem 5. Evaluate each of the following exactly.

1. $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

   Answer:

   If we let $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, then we know that

   $$\begin{gathered} \cos(\theta) = -\sqrt{3}/2 \\ 0 \le \theta \le \pi. \end{gathered}$$

   We know $\cos(\pi/6) = /sqrt3/2$, so the angle $\pi/6$ gives us a point

   $$(\cos(\pi/6), \sin(\pi/6)) = (\sqrt{3}/2, 1/2)$$

   on the unit circle with x-coordinate $\sqrt{3}/2$. See Figure 2. Reflecting this point through the y-axis gives us a point with x-coordinate $-\sqrt{3}/2$. Hence the angle θ we want is the one marked in the diagram, which is $\theta = \pi - \pi/6 = 5\pi/6$.

2. $\tan(\cos^{-1}(x))$.

   Answer:

   Let $\theta = \cos^{-1}(x)$, so $\cos(\theta) = x$. Label the angle θ in the reference triangle and label the sides to reflect $\cos(\theta) = x$. See Figure 3. We can find the

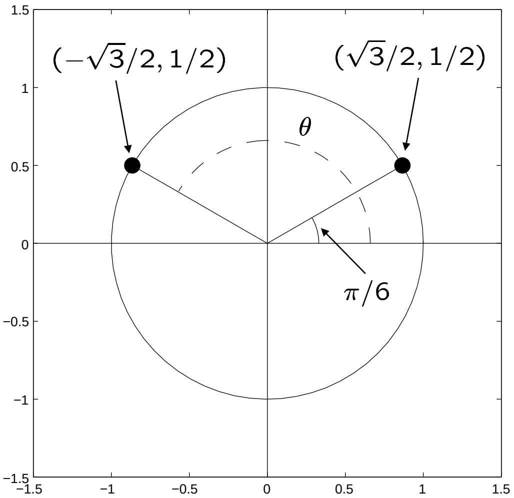

Figure 2: Figure for Problem 5, part 1

length b of the opposite side by the Pythagorean theorem: $x^2 + b^2 = 1^2$, so $b = \sqrt{1 - x^2}$. Since $\tan\theta$ is given by the opposite side over the adjacent side, we have

$$\tan(\cos^{-1}(x)) = \frac{\sqrt{1 - x^2}}{x}.$$

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40 pts.

Problem 6. **Use the definition of the derivative as the limit of a quotient** to find $f'(x)$, where $f(x) = 5x^2 + x$. (Sorry, no credit for finding it by another method.)

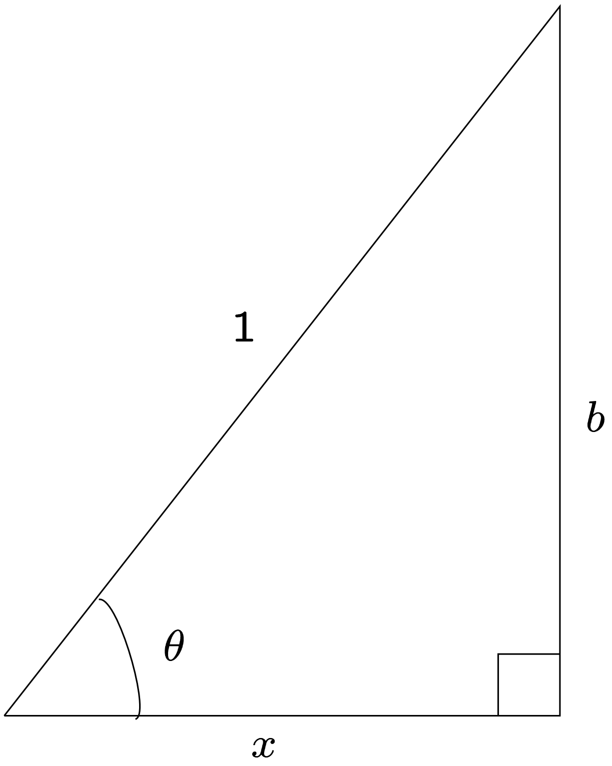

Figure 3: The Figure for Problem 5, part 2

Answer:
We compute as follows:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5(x+\Delta x)^2 + (x+\Delta x) - (5x^2 + x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5(x^2 + 2x\Delta x + (\Delta x)^2) + x + \Delta x - 5x^2 - x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5x^2 + 10x\Delta x + 5(\Delta x)^2 + x + \Delta x - 5x^2 - x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{10x\Delta x + 5(\Delta x)^2 + \Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(10x + 5\Delta x + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (10x + 1 + 5\Delta x) \\
&= 10x + 1 + 5(0) \\
&= 10x + 1.
\end{aligned}$$

Thus, we conclude that $f'(x) = 10x + 1$.