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**EXAM**

Exam #3

Math 1351-012, Fall 2002

November 22, 2002

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**ANSWERS**



40 pts.

**Problem 1.** Find the absolute maximum and absolute minimum values of the function  $f(x) = 3x^4 - 8x^3 + 2$  on the interval  $[-1, 3]$ .

*Answer:*

To find the critical numbers, we calculate

$$f'(x) = 12x^3 - 24x^2 = 12x^2(x - 2).$$

Thus,  $f'(x) = 0$  at  $x = 0$  and  $x = 2$ . Both of these points are in the interval  $[-1, 3]$ . Since the interval is closed and bounded, we can find the absolute maximum and minimum by evaluating the function at the endpoints and at the critical points in the interior of the interval. Using a calculator, we get

$$\begin{aligned}f(-1) &= 13 \\f(0) &= 2 \\f(2) &= -14 \\f(3) &= 29.\end{aligned}$$

Thus, the absolute maximum of  $f$  on  $[-1, 3]$  is 29, which occurs at the endpoint  $x = 3$ , and the absolute minimum is  $-14$ , which occurs at the critical point  $x = 2$ .

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40 pts.

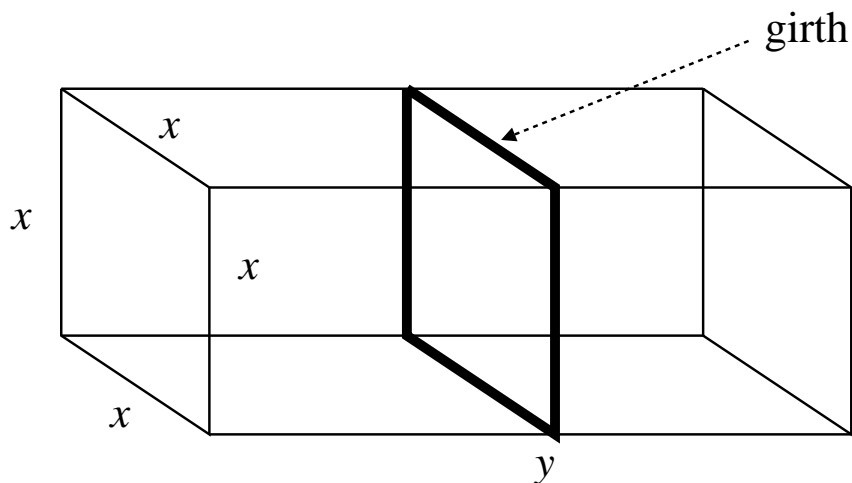
**Problem 2.** A shipping company will accept a package if the length plus the girth (distance around) is less than or equal to 180 inches. What are the dimensions of the rectangular package with square ends that can be shipped with this company and has the largest possible volume.

Be sure to show what interval you are maximizing the function over, and to justify that you have really found the max, not just a critical point.

[If the picture is unclear, ask about it.]

*Answer:*

Labeling the sides of the square ends as  $x$  and the length as  $y$ , the picture looks like this.



The girth is  $4x$  and the length is  $y$ , so we must have

$$4x + y = 180. \tag{1}$$

(It seems clear we should use all of the 180 inches we're allowed.) The volume of the box is

$$V = x^2y \tag{2}$$

Thus, we want to maximize (2) subject to the constraint (1).

If we solve (1) for  $y$ , we have

$$y = 180 - 4x. \tag{3}$$

Plugging this into (2) gives

$$V = x^2(180 - 4x) = 180x^2 - 4x^3 \tag{4}$$

for the function we want to maximize. What interval do we want to maximize this function over? We must have  $x \geq 0$ , since it is the length of a side of the package. For the same reason, we must have  $0 \leq y = 180 - 4x$ . The solution of this inequality is  $x \leq 45$ . Thus, we want to maximize (4) over the interval  $[0, 45]$ .

Taking the derivative of  $V$ , we have

$$V' = 360x - 12x^2 = 12x(30 - x),$$

so the critical points are  $x = 0$  (not really relevant) and  $x = 30$ . To find the absolute max, we have to check the critical points in the interval and the endpoints. Thus, using (4),

$$\begin{aligned} V(0) &= 0 \\ V(30) &= 54,000 \\ V(45) &= 0. \end{aligned}$$

So, we get absolute max at  $x = 30$ . To find the corresponding value of  $y$ , plug into (3), so  $y = 60$ . Thus, the dimensions for the maximum volume are

$$\boxed{x = 30, \quad y = 60}.$$

50 pts.

**Problem 3.** Consider the function  $f(x) = 3x^4 - 24x^3 + 1$ .

- A. Find all the critical points. Construct the sign table for  $f'(x)$ . Determine where  $f(x)$  is increasing and decreasing. Determine if each critical point is a relative max, a relative min, or neither.

*Answer:*

We have  $f'(x) = 12x^3 - 72x^2 = 12x^2(x - 6)$ , so the critical numbers are  $x = 0, 6$ . The sign table is shown below (below the line we show the sign of each factor, used to find the sign of  $f'$ ).

$f'$	-	0	-	0	+
←————— —————→					
		0		6	
$12x^2$	+		+		+
$(x - 6)$	-		-		+

Thus we conclude that  $f$  is decreasing on the interval  $(-\infty, 6)$  and increasing on the interval  $(6, \infty)$ . From the first derivative test, the critical point at  $x = 0$  is neither a relative max nor a relative min, while the critical point at  $x = 6$  is a relative min.

- B. Find the sign table for  $f''(x)$ . Find all inflection points. Determine where the graph  $y = f(x)$  is concave up and concave down.

*Answer:*

We have  $f''(x) = 36x^2 - 144x = 36x(x - 4)$ , which has zeros at  $x = 0$  and  $x = 4$ . If we construct the sign table we get

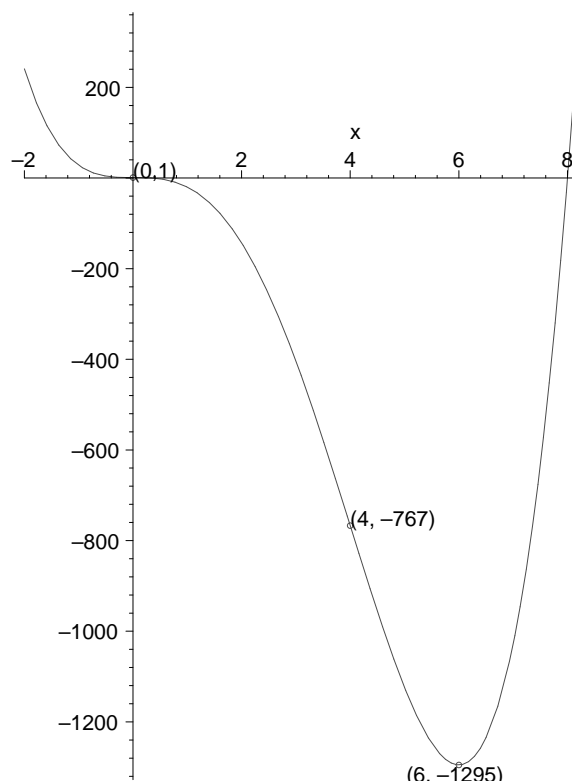
$f''$	+	0	-	0	+
←————— —————→					
		0		4	
$36x$	-		+		+
$(x - 4)$	-		-		+

Thus,  $f$  is concave up on  $(-\infty, 0)$  and  $(4, \infty)$  and concave down on  $(0, 4)$ . Both  $x = 0$  and  $x = 4$  are inflection points.

C. Sketch the graph  $y = f(x)$ , plotting all critical points and inflection points.

*Answer:*

Here is a computer sketch of the graph, with the critical points and inflection points marked.



60 pts.

**Problem 4.** Consider the function

$$f(x) = \frac{1}{x^2 - 4}.$$

To save time, I'll give you the formulas

$$f'(x) = -\frac{2x}{(x^2 - 4)^2}, \quad f''(x) = \frac{6x^2 + 8}{(x^2 - 4)^3}.$$

- A. Find all vertical and horizontal asymptotes for the graph  $y = f(x)$ .

*Answer:*

The denominator of  $f(x)$  is zero at  $x = -2$  and  $x = 2$ , and the numerator is not zero at these points, so we have vertical asymptotes  $x = -2$  and  $x = 2$ . To find the horizontal asymptotes, we consider the limit of  $f(x)$  as  $x \rightarrow \pm\infty$ :

$$\begin{aligned}\lim_{x \rightarrow \pm\infty} f(x) &= \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 4} \\ &= \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 1} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x^2}}{1 - \frac{4}{x^2}} \\ &= \frac{0}{1 - 0} = 0,\end{aligned}$$

so we have a horizontal asymptote  $y = 0$ .

- B. Find all the critical points. Construct the sign table for  $f'(x)$ . Determine if  $f(x)$  is increasing and decreasing. Determine if each critical point is a relative max, a relative min, or neither.

*Answer:*

From the formula for the first derivative, we see that the only zero of  $f'$  is  $x = 0$ , so that is the only critical number. However,  $f'$  is undefined for  $x = \pm 2$  (like the function  $f$ ), so the sign could change at these points. Thus, when making the sign table, we need division points at  $x = -2, 0, 2$ . The sign table is as follows (the symbol  $\downarrow$  indicates a vertical asymptote at that point):

$f'$	+	$\downarrow$	+	0	-	$\downarrow$	-
$(x^2 - 4)^2$	+	$\downarrow$	+	0	+	$\downarrow$	+
$x$	-	$\downarrow$	-	0	+	$\downarrow$	+
-2	-	$\downarrow$	-	0	-	$\downarrow$	-

Thus, by the first derivative test, the critical point  $x = 0$  is a relative max. The function is increasing on the intervals  $(-\infty, -2)$  and  $(-2, 0)$ ; and decreasing on  $(0, 2)$  and  $(2, \infty)$ .

- C. Find the sign table for  $f''(x)$ . Find all inflection points. Determine where the graph  $y = f(x)$  is concave up and concave down.

*Answer:*

From the formula for  $f''$ , we see that  $f''$  has no zeros, since the numerator is never zero, so there are no inflection points. However,  $f''$  is undefined at  $x = \pm 2$  (like the function  $f$ ), so we will have to use  $x = -2, 2$  as the division points in the sign table. The sign table is as follows:

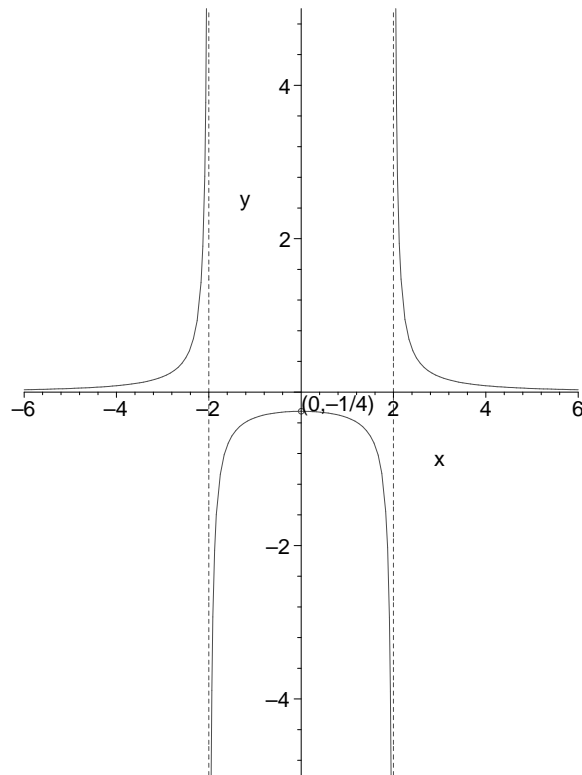
$f''$	+	↓	-	↓	+
	←----- ----- -----→				
		-2		2	
$(x - 2)^3$	-		-		+
$(x + 2)^3$	-		+		+
$(6x^2 + 8)$	+		+		+

From this we conclude that the graph  $y = f(x)$  is concave up on the intervals  $(-\infty, -2)$  and  $(2, \infty)$ ; and concave down on the interval  $(-2, 2)$ .

- D. Sketch the graph  $y = f(x)$ , plotting all critical points, inflection points, and asymptotes.

*Answer:*

Here is a computer sketch of the graph. The vertical asymptotes are drawn as dotted lines and the critical point at  $(0, -2/4)$  is marked. The horizontal asymptote is the  $x$ -axis, which is not specially marked.



40 pts.

**Problem 5.** In each part, find the limit.

A.

$$\lim_{x \rightarrow 0} \frac{\ln(x^2 + 1)}{x^2}$$

*Answer:*

This has indeterminate form  $0/0$ . Applying L'Hôpital's rule, we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(x^2 + 1)}{x^2} &= \frac{2x}{2x} \\ &= \lim_{x \rightarrow 0} \frac{1}{x^2 + 1} \\ &= \frac{1}{0 + 1} = 1. \end{aligned}$$

B.

$$\lim_{x \rightarrow 0} \frac{\cos(3x) - 1}{x^3}$$

*Answer:*

This has the indeterminate form  $0/0$ . By L'Hôpital's rule, we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos(3x) - 1}{x^3} &= \lim_{x \rightarrow 0} \frac{-3 \sin(3x)}{3x^2} \\ &= - \lim_{x \rightarrow 0} \frac{\sin(3x)}{x^2} && \text{still } \frac{0}{0} \\ &= - \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{2x} \\ &= - \frac{3}{2} \lim_{x \rightarrow 0} \frac{\cos(3x)}{x}, \end{aligned}$$

however, as  $x \rightarrow 0$ ,  $\cos(3x) \rightarrow 1$ , thus the limit in the last line behaves similarly to  $1/x$  as  $x \rightarrow 0$ , so the limit does not exist.

C.

$$\lim_{x \rightarrow \infty} \frac{[\ln(x)]^2}{x}$$

*Answer:*

This has the indeterminate form  $\infty/\infty$ . We have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{[\ln(x)]^2}{x} &= \lim_{x \rightarrow \infty} \frac{2 \ln(x)}{1} && \text{by L'H's rule} \\ &= 2 \lim_{x \rightarrow \infty} \frac{\ln(x)}{x} && \text{has form } \frac{\infty}{\infty} \\ &= 2 \lim_{x \rightarrow \infty} \frac{1/x}{1} && \text{by L'H's rule} \\ &= 2 \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= 2(0) = 0. \end{aligned}$$

D.

$$\lim_{x \rightarrow 0^+} x \ln(x)$$

*Answer:*

This has the indeterminate form  $0 \cdot \infty$ . We must rewrite the expression as a quotient so we get the form  $0/0$  or  $\infty/\infty$  and then use L'Hôpital's rule. So, we have

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln(x) &= \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} && \text{has the form } \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} && \text{by L'H's rule} \\ &= \lim_{x \rightarrow 0^+} \frac{1/x \cdot -x^2}{-1/x^2 \cdot -x^2} \\ &= \lim_{x \rightarrow 0^+} (-x) = 0.\end{aligned}$$

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