EXAM

Exam #1

Math 1351-012, Fall 2002

October 2, 2002

ANSWERS

90 pts. **Problem 1**. Figure 1 shows the graph of a function f. In each part, find the right-hand limit, left-hand limit and two-sided limit as x approaches a; find the value $f(a)$; and determine if the function is continuous at a. Justify your answers.

A. $a = -2$.

Answer:

We have

$$\lim_{x \to -2^-} f(x) = 5$$
$$\lim_{x \to -2^+} f(x) = 5$$

and so

$$\lim_{x \to -2} f(x) = 5.$$

From the graph, $f(-2) = 1$. The function is not continuous at -2, since

$$\lim_{x \to -2} f(x) \neq f(-2).$$

B. $a = 0$.

Answer:

We have

$$\lim_{x \to 0^-} f(x) = 3$$
$$\lim_{x \to 0^+} f(x) = 3$$

and so

$$\lim_{x \to 0} f(x) = 3.$$

We also have $f(0) = 3$, so the function is continuous at $x = 0$.

C. $a = 2$.

Answer:

We have

$$\lim_{x \to 2^-} f(x) = 1$$
$$\lim_{x \to 2^+} f(x) = 4$$

and so

$$\lim_{x \to 2} f(x), \quad \text{does not exist.}$$

The value of $f(2)$ is $f(2) = 2$. The function is not continuous at $x = 2$, because $\lim_{x \to 2} f(x)$ does not exist.

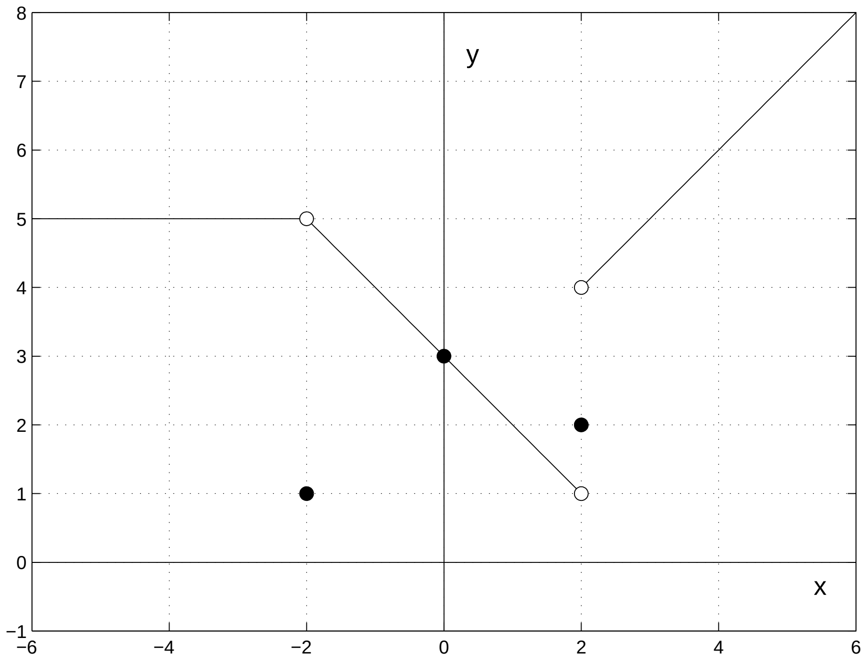

Figure 1: The function for Problem 1

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50 pts. **Problem 2**. Consider the function

$$f(x) = \begin{cases} x^2+1, & x<0 \\ 2, & x=0 \\ x+1, & 0<x\leq 2 \qquad \text{(note correction)} \\ x^2-1, & x>2. \end{cases}$$

Find the suspicious points and determine if f is continuous at each suspicious point. Explain your answers.

Answer:
The suspicious points are $x=0$ and $x=2$, because the formulas change at these points. At $x=2$ we have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2+1) = 0^2+1 = 1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 0+1 = 1$$

and hence

$$\lim_{x\to 0} f(x) = 1.$$

However, $f(0) = 2$, so the function is *not* continuous at $x = 0$, because

$$\lim_{x \to 0} f(x) \neq f(0).$$

For the suspicious point $x = 2$, we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 1) = 2 + 1 = 3$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 1) = (2)^2 - 1 = 3$$

and so

$$\lim_{x \to 2} f(x) = 3.$$

We also have $f(2) = 2 + 1 = 3$, so the function is continuous at $x = 2$, since

$$\lim_{x \to 2} f(x) = f(2).$$

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80 pts.

Problem 3. In each part, find the limit (if it exists).

A.

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 2}$$

Answer:

The denominator does not vanish at $x = 1$, so we can just use the quotient rule for limits and the limits of polynomials. Thus,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 2} = \frac{1^2 + 1 - 2}{1 - 2} = 0.$$

B.

$$\lim_{x \to 0} \frac{x}{\sin(5x)}$$

Answer:

We calculate as follows:

$$\begin{aligned}
\lim_{x \to 0} \frac{x}{\sin(5x)} &= \lim_{x \to 0} \frac{5x}{5\sin(5x)} \\
&= \frac{1}{5} \lim_{x \to 0} \frac{5x}{\sin(5x)} \\
&= \frac{1}{5} \lim_{x \to 0} \frac{1}{\frac{\sin(5x)}{5x}} \\
&= \frac{1}{5} \frac{1}{1} \\
&= \frac{1}{5},
\end{aligned}$$

using the limit

$$(*) \qquad \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

from class.

C.

$$\lim_{x \to 0} \frac{\tan(x)}{\tan(2x)}$$

Answer:

Putting in lots of detail, we can calculate as follows:

$$\begin{aligned}
\lim_{x \to 0} \frac{\tan(x)}{\tan(2x)} &= \lim_{x \to 0} \frac{\frac{\sin(x)}{\cos(x)}}{\frac{\sin(2x)}{\cos(2x)}} \\
&= \lim_{x \to 0} \frac{\cos(2x)}{\sin(2x)} \frac{\sin(x)}{\cos(x)} \\
&= \lim_{x \to 0} \frac{\cos(2x)}{\cos(x)} \frac{\sin(x)}{\sin(2x)} \\
&= \lim_{x \to 0} \frac{\cos(2x)}{\cos(x)} \frac{\sin(x)}{\sin(2x)} \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x \to 0} \frac{\cos(2x)}{\cos(x)} \frac{\frac{\sin(x)}{x}}{\frac{\sin(2x)}{x}} \\
&= \lim_{x \to 0} \frac{\cos(2x)}{\cos(x)} \frac{\frac{\sin(x)}{x}}{2\frac{\sin(x)}{2x}} \\
&= \frac{1}{1} \frac{1}{2(1)} \\
&= \frac{1}{2}.
\end{aligned}$$

Using the limit $(*)$ and the fact that cos is a continuous function with $\cos(0) = 1$.

D.

$$\lim_{x \to 2} \frac{\sqrt{2x+5} - 3}{x - 2}$$

Answer:
We make use of the identity $(a-b)(a+b) = a^2 - b^2$ and calculate as follows:

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2} &= \lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2}\,\frac{\sqrt{2x+5}+3}{\sqrt{2x+5}+3} \\
&= \lim_{x\to 2} \frac{(\sqrt{2x+5})^2 - 3^2}{(x-2)(\sqrt{2x+5}+3)} \\
&= \lim_{x\to 2} \frac{2x+5-9}{(x-2)(\sqrt{2x+5}+3)} \\
&= \lim_{x\to 2} \frac{2x-4}{(x-2)(\sqrt{2x+5}+3)} \\
&= \lim_{x\to 2} \frac{2(x-2)}{(x-2)(\sqrt{2x+5}+3} \\
&= \lim_{x\to 2} \frac{2}{\sqrt{2x+5}+3} \\
&= \frac{2}{\sqrt{2(2)+5}+3} \\
&= \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

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40 pts.

Problem 4. Solve the following equation:

$$\ln(x) + \ln(3x+2) = 0.$$

Answer:
We solve the equation by the following sequence of steps:

$$\begin{aligned}
(*) \qquad \ln(x) + \ln(3x+2) &= 0 & & \\
\ln(x(3x+2)) &= 0 & & \text{product rule for logs} \\
e^{\ln(x(3x+2))} &= e^0 & & \text{take exponential of both sides} \\
x(3x+2) &= 1 & & \text{inverse relations, } e^0 = 1 \\
3x^2 + 2x &= 1 & & \\
3x^2 + 2x - 1 &= 0 & & \\
(3x-1)(x+1) &= 0 & &
\end{aligned}$$

The solutions of the last equation are $x = -1$ and $x = 1/3$. However, if we plug $x = -1$ into the original equation $(*)$, the first term becomes $\ln(-1)$, which is undefined. Thus, $x = -1$ is *not* a solution of the original equation $(*)$. If we plug $x = 1/3$ into $(*)$, both of the expressions inside the logs are positive, and so defined. We conclude that the only solution of $(*)$ is $x = 1/3$.

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40 pts. **Problem 5**. Evaluate each of the following exactly.

1. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

   Answer:

   If we let $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, then we know that

   $$\cos(\theta) = -\sqrt{2}/2$$
   $$0 \leq \theta \leq \pi.$$

   We know $\cos(\pi/4) = \sqrt{2}/2$, so the angle $\pi/4$ gives us a point

   $$(\cos(\pi/4), \sin(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$$

   on the unit circle with x-coordinate $\sqrt{2}/2$. See Figure 2. Reflecting this point through the y-axis gives us a point with x-coordinate $-\sqrt{2}/2$. Hence the angle θ we want is the one marked in the diagram, which is $\theta = \pi - \pi/4 = 3\pi/4$.

2. $\sin(\tan^{-1}(x))$.

   Answer:

   Let $\theta = \tan^{-1}(x)$, so $\tan(\theta) = x$. Label the angle θ in the reference triangle and label the sides to reflect $\tan(\theta) = x$. See Figure 3. We can find the length c of the hypotenuse by the Pythagorean theorem: $c^2 = 1^2 + x^2$, so $c = \sqrt{1 + x^2}$. Since $\sin(\theta)$ is the opposite side over the hypotenuse, we have

   $$\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1 + x^2}}.$$

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40 pts. **Problem 6**. **Use the definition of the derivative as the limit of a quotient** to find $f'(x)$, where $f(x) = x^2 + 3x$. (Sorry, no credit for finding it by another method.)

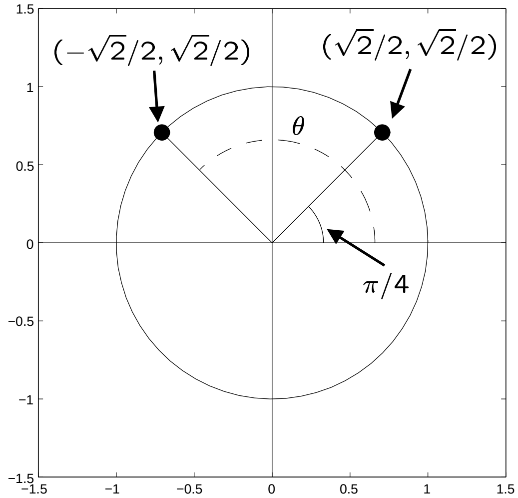

Figure 2: Figure for Problem 5, part 1

Answer:
We compute as follows:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 3(x+\Delta x) - (x^2+3x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x - x^2 - 3x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 3\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x + 3)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x + 3) \\
&= 2x + 0 + 3 \\
&= 2x + 3.
\end{aligned}$$

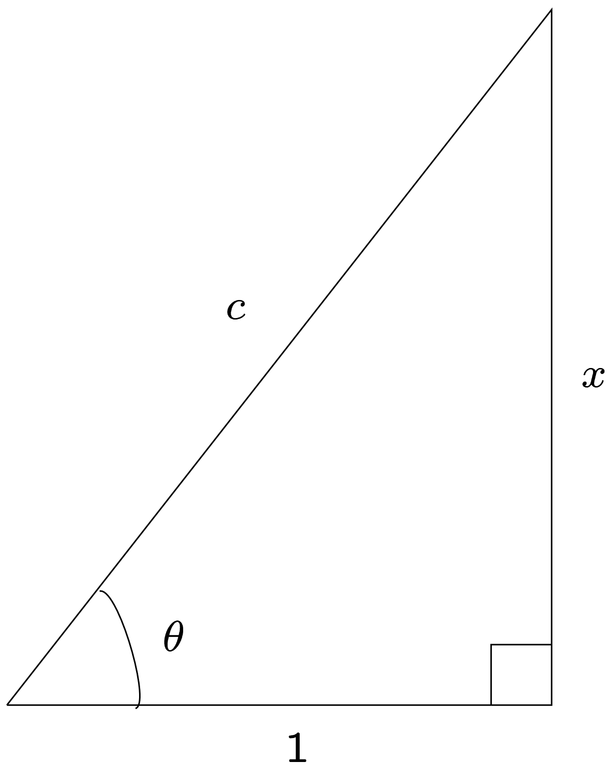

Figure 3: The Figure for Problem 5, part 2