# NOTES ON SUP'S, INF'S AND SEQUENCES

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The purpose of these notes is to briefly review some material on sup's, inf's and sequences from undergraduate real analysis, to give an introduction to using these concepts in the extended real numbers, and to give an exposition of lim sup and lim inf.

## 1. Sup's and Inf's in the Real Numbers

I'll use the symbol $\mathbb{R}$ to denote the set of real numbers.

Let $A$ be a nonempty subset of $\mathbb{R}$. A number $u$ is **an upper bound for** $A$ if $a \leq u$ for all $a \in A$. We say that $A$ is **bounded above** if it has an upper bound.

Note that saying that a number $t$ is *not* an upper bound for $A$ is equivalent to the statement "There is some $a \in A$ such that $t < a$."

A number $s$ is called the **supremum of** $A$ (sup) if it is the least upper bound of $A$, i.e., $s$ is an upper bound for $A$ and if $u$ is an upper bound for $A$ then $s \leq u$. We denote the supremum of $A$ by $\sup(A)$. (A little thought shows there can be at most one number that satisfies the definition of sup.)

As you recall, the real numbers are constructed by "filling in the holes" in the rational numbers. One way of saying that the holes have all been filled is the Completeness Axiom for the Real Numbers, which is stated as follows.

**Completeness Axiom for the Real Numbers.** *If $A$ is a nonempty subset of $\mathbb{R}$ that is bounded above then $A$ has a supremum.*

We have similar concepts when working on the other side of $A$. Again, let $A$ be a nonempty subset of $\mathbb{R}$. A number $\ell$ is a **lower bound for** $A$ if $\ell \leq a$ for all $a \in A$. We say $A$ is **bounded below** if it has a lower bound.

A number $i$ is the **infimum of** $A$ if it is the greatest lower bound of $A$, i.e., $i$ is a lower bound and if $\ell$ is a lower bound for $A$, then $\ell \leq i$.

It is not necessary to add an axiom about inf's to the definition of the real numbers, since the existence of inf's can be deduced from the existence of sup's by what I like to call "The Reflection Trick." Here we set

$$-A = \{-a \mid a \in A\}.$$

**Proposition 1.1. [Reflection Trick]** *Let $A$ be a nonempty subset of $\mathbb{R}$.*

(1) *If $A$ is bounded below, $-A$ is bounded above and*
$$\inf(A) = -\sup(-A).$$

(2) *If $A$ is bounded above, $-A$ is bounded below and*
$$\sup(A) = -\inf(-A).$$

**Exercise 1.2.** Prove the last Proposition.

The following Propositions are frequently used in proofs involving sup and inf.

**Proposition 1.3.** *Let $A$ be a nonempty subset of $\mathbb{R}$.*

(1) *Suppose that $A$ is bounded above. Then $\alpha < \sup(A)$ if and only if $\alpha < a$ for some $a \in A$.*
(2) *Suppose that $A$ is bounded below. Then $\inf(A) < \beta$ if and only if $a < \beta$ for some $a \in A$.*

*Proof.* To prove the first statement, assume first that $\alpha < \sup(A)$. Since $\alpha$ is less than the least upper bound of $A$, $\alpha$ is not an upper bound for $A$. Thus, there is some $a \in A$ so that $\alpha < a$.

For the second part of the proof, assume that $\alpha < a$ for some $a \in A$. Then $\alpha < a \leq \sup(A)$ (since $\sup(A)$ is an upper bound), so $\alpha < \sup(A)$.

The proof of the second statement is similar and is left as an exercise. □

**Proposition 1.4.** *Suppose that $A$ is a nonempty subset of $\mathbb{R}$. Then* $\inf(A) \leq \sup(A)$

*Proof.* Since $A$ is nonempty, we can find some $a \in A$. We have $\inf(A) \leq a$, since $\inf(A)$ is a lower bound and we have $a \leq \sup(A)$ since $\sup(A)$ is an upper bound. Hence $\inf(A) \leq a \leq \sup(A)$. □

**Proposition 1.5.** *Suppose that $A \subseteq B \subseteq \mathbb{R}$ and $A \neq \emptyset$. Then*

(1) *If $B$ is bounded above,* $\sup(A) \leq \sup(B)$.
(2) *If $B$ is bounded below,* $\inf(A) \geq \inf(B)$.

*Proof.* If $B$ is bounded above, $\sup(B)$ exists. Since $\sup(B)$ is an upper bound for $B$, we have $b \leq \sup(B)$ for all $b \in B$. In particular, we have $a \leq \sup(B)$ for all $a \in A$. Thus, $\sup(B)$ is an upper bound for $A$ (so $A$ is bounded above) and we must have $\sup(A) \leq \sup(B)$ since $\sup(A)$ is the *least* upper bound of $A$.

The proof of the second statement is similar. If $B$ is bounded below, $\inf(B)$ exists. We have $\inf(B) \leq b$ for all $b \in B$. In particular, we have $\inf(B) \leq a$ for all $a \in A$, so $\inf(B)$ is a lower bound for $A$ (so $A$ is bounded below). We must have $\inf(B) \leq \inf(A)$, since $\inf(A)$ is the *greatest* lower bound of $A$. □

## 2. Sequences of Reals Numbers

In this section, we will give a brief review of the basic material on sequences of real numbers typically presented in an undergraduate analysis course. The statements labeled "Fact" should be familiar to you—if the proof does not immediately come to mind, consider the statement as an exercise.

2.1. **Convergence of Sequences.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. A sequence of real numbers is a function $\mathbb{N} \to \mathbb{R}$. It is traditional to use subscripts (rather than functional notation) to describe the function. Thus the sequence $\{a_n\}_{n=1}^{\infty}$ is the same thing as the function $n \mapsto a_n$.

The first and most important concept about sequences is convergence.

**Definition 2.1.** The sequence $\{a_n\} \subseteq \mathbb{R}$ **converges to** $a \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n \geq N$.

**Fact 2.2.** *A sequence converges to at most one real number.*

The statement that $\{a_n\}$ converges to $a$ is written in symbols as

$$\lim_{n\to\infty} a_n = a, \quad \text{or} \quad a_n \to a.$$

**Fact 2.3.** *If $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$.*

We will prove the corresponding statement for products in a moment. In order to do so, we need that following fact.

**Fact 2.4.** *A convergent sequence is bounded, i.e., if $\{a_n\}$ converges, there is some number $M$ so that $|a_n| \leq M$ for all $n \in N$.*

We also need the following famous statement.

**Triangle Inequality.** *If $x$ and $y$ are real numbers,*

$$\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|.$$

We're now ready to prove the following Proposition.

**Proposition 2.5.** *If $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.*

*Proof.* The basic idea of the proof is based on the following inequalities.

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

Thus, we have

$$|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b|. \tag{2.1}$$

Since $\{b_n\}$ is convergent, there is some number $M > 0$ so that $|b_n| \leq M$ for all $n$.

Let $\varepsilon > 0$ be given. Since $a_n \to a$, there is some $N_1 \in \mathbb{N}$ so that

$$|a_n - a| < \varepsilon/M \tag{2.2}$$

for all $n \geq N_1$. Since $b_n \to b$, there is some $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{1 + |a|} \tag{2.3}$$

for $n \geq N_2$.

Set $N = \max(N_1, N_2)$. If $n \geq N$ both (2.2) and (2.3) hold. Thus, if $n \geq N$, (2.1) gives us

$$\begin{aligned} |a_n b_n - ab| &\leq |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \\ &< M\frac{\varepsilon}{M} + |a|\frac{\varepsilon}{1 + |a|} \\ &< \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

since $0 \leq |a|/(1 + |a|) < 1$.

Thus, we have

$$|a_n b_n - ab| < 2\varepsilon$$

for $n \geq N$. This completes the proof. □

*Remark.* You might object that we were supposed to get $|a_n b_n - ab| < \varepsilon$ instead of less that $2\varepsilon$. But this is okay. Look at it this way: you hand me an arbitrary $\eta > 0$ and tell me to make the desired quantity $|a_n b_n - ab|$ less than $\eta$. I just choose some number $\varepsilon$ so small that $2\varepsilon < \eta$. I proceed with my proof to get $|a_n b_n - ab| < 2\varepsilon < \eta$, so I've met your challenge.

In general it's okay to show that given $\varepsilon > 0$ you can make the desired quantity less than $K\varepsilon$, for some constant $K$. Here "constant" means that $K$ does not depend on the choice of $\varepsilon$.

It is convenient to make a few definitions that allow us to restate the definition of convergence and related concepts concisely.

We say that the sequence $\{a_n\}$ is **eventually in** a set $E$ if there is an $N \in \mathbb{N}$ so that $a_n \in E$ for all $n \geq N$.

**Proposition 2.6.** *Suppose that $\{a_n\}$ is eventually in $E_1$ and eventually in $E_2$. Then $\{a_n\}$ is eventually in $E_1 \cap E_2$.*

*Proof.* Since the sequence is eventually in $E_1$, there is some $N_1 \in \mathbb{N}$ so that $a_n \in E_1$ for $n \geq N_1$. Similarly, there is some $N_2$ so that $a_n \in E_2$ for $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then, $a_n \in E_1 \cap E_2$ for $n \geq N$. □

Let $a$ be a point in $\mathbb{R}$. A set $U \subseteq \mathbb{R}$ is a **neighborhood of** $a$ if there is some $\varepsilon > 0$ so that $(a - \varepsilon, a + \varepsilon) \subseteq U$.

**Fact 2.7.** *Let $\{a_n\} \subseteq \mathbb{R}$. Then, the following statements are equivalent.*

1. $a_n \to a$.
2. *For every $\varepsilon > 0$, the sequence $\{a_n\}$ is eventually in the interval $(a-\varepsilon, a+\varepsilon)$.*
3. *The sequence $\{a_n\}$ is eventually in every neighborhood of $a$.*

**Proposition 2.8.** *Suppose that $a_n \to a$ and $b_n \to b$ and that there is some $M \in \mathbb{N}$ so that $a_n \leq b_n$ for $n \geq M$. Then $a \leq b$.*

*Proof.* The proof is by contradiction. Suppose, for a contradiction, that $b < a$.

Choose some number $\varepsilon > 0$ so that $b + \varepsilon < a - \varepsilon$ ( any $\varepsilon < (a - b)/2$ will do). Since $a_n \to a$, there is some $N_1 \in \mathbb{N}$ so that $a_n \in (a-\varepsilon, a+\varepsilon)$ for $n \geq N_1$. Similarly, there is an $N_2 \in \mathbb{N}$ so that $b_n \in (b - \varepsilon, b + \varepsilon)$ for $n \geq N_2$.

Set $N = \max(M, N_1, N_2)$. On one hand, since $N \geq M$ we have $a_N \leq b_N$. On the other hand, we have

$$b_N < b + \varepsilon < a - \varepsilon < a_N,$$

so $b_N < a_N$. This contradiction shows that our original assumption that $b < a$ must be wrong. □

**Fact 2.9.** *Suppose that $a_n \to a$ and that eventually $\alpha \leq a_n \leq \beta$. Then $\alpha \leq a \leq \beta$.*

A sequence $\{a_n\}$ is called **increasing** if $a_n \leq a_{n+1}$ for all $n$, i.e.,

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots .$$

Similarly, the sequence is called **decreasing** if $a_n \geq a_{n+1}$ for all $n$.

The following Proposition relates convergence and sup's.

**Proposition 2.10.** *Let $\{a_n\}$ be a sequence of real numbers.*

1. *if $\{a_n\}$ is increasing and bounded above, then the sequence is convergent and*
$$\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}.$$
2. *if $\{a_n\}$ is decreasing and bounded below, then the sequence is convergent and*
$$\lim_{n\to\infty} a_n = \inf\{a_n \mid n \in \mathbb{N}\}.$$

*Proof.* We'll prove the first part of the Proposition and leave the second part as an exercise.

We're assuming that $\{a_n\}$ is bounded above, meaning that the set of values of the sequence $\{a_n \mid n \in \mathbb{N}\}$ is bounded above. Thus, $s = \sup\{a_n \mid n \in \mathbb{N}\}$ exists.

Let $\varepsilon > 0$ be arbitrary. Since $s - \varepsilon < s$, $s - \varepsilon$ is not an upper bound for the values of the sequence, so there is some $N$ so that $s - \varepsilon < a_N$. Since the sequence is increasing $a_n \geq a_N$ for $n \geq N$. Thus, for $n \geq N$ we have

$$s - \varepsilon < a_N \leq a_n \leq s.$$

Thus, $|a_n - a| < \varepsilon$ for $n \geq N$. This shows that $a_n \to s$. □

2.2. **Cluster Points and Subsequences.** If $\{a_n\}$ is a sequence, we say that $\{a_n\}$ is **frequently in** a set $E$ if, for every $N \in \mathbb{N}$, there is an $n \geq N$ so that $a_n \in E$. Another way to say it is that infinitely many terms of the sequence are in $E$ (but there may also be infinitely many terms that are not in $E$).

If $\{a_n\}$ is a sequence of real numbers, we say $p \in \mathbb{R}$ is a **cluster point of** $\{a_n\}$ if the sequence is frequently in every neighborhood of $p$. Equivalently, every neighborhood of $p$ contains infinitely many terms of the sequence.

If we have a strictly increasing sequence of natural numbers

$$n_1 < n_2 < n_3 < \cdots,$$

the sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence of** $\{a_n\}$.

**Fact 2.11.** *If $a_n \to a$, then every subsequence of $\{a_n\}$ converges to $a$.*

**Fact 2.12.** *A subsequence of a subsequence of $\{a_n\}$ is a subsequence of $\{a_n\}$.*

The relationship between cluster points and subsequences is given by the following proposition.

**Proposition 2.13.** *Let $\{a_n\} \subseteq \mathbb{R}$ be a sequence. A point $p \in \mathbb{R}$ is a cluster point of $\{a_n\}$ if and only if there is a subsequence $\{a_{n_k}\}$ of the original sequence so that $a_{n_k} \to p$.*

*Proof.* For the first part of the proof, suppose that $p$ is a cluster point of $\{a_n\}$. Then the sequence is frequently in every neighborhood of $p$. Since $(p-1, p+1)$ is a neighborhood of $p$, there is some $n_1$ such that $a_{n_1} \in (p-1, p+1)$.

There are infinitely many terms of the sequence in the neighborhood $(p-1/2, p+1/2)$ of $p$, so we can find some $n_2 > n_1$ so that $a_{n_2} \in (p-1/2, p+1/2)$. Continuing in this way (technically an inductive construction), we get a sequence

$$n_1 < n_2 < n_3 < \cdots$$

of natural numbers such that $a_{n_k} \in (p - 1/k, p + 1/k)$, equivalent, $|a_{n_k} - p| < 1/k$.

It's fairly clear that the subsequence $\{a_{n_k}\}$ converges to $p$. If $k \geq K$, then $(p - 1/k, p + 1/k) \subseteq (p - 1/K, p + 1/K)$, so $|a_{n_k} - p| < 1/K$ for $k \geq K$. Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ so that $1/K < \varepsilon$. Then $|a_{n_k} - p| < 1/K < \varepsilon$ for $k \geq K$. Thus, $a_{n_k} \to p$.

For the converse, assume that there is a subsequence $\{a_{n_k}\}$ that converges to $p$. Suppose that $U$ is a neighborhood of $p$. Let $N$ be a natural number. Since $a_{n_k} \to p$, $a_{n_k}$ is eventually in $U$. Thus, there is a $K$ so that $a_{n_k} \in U$ for $k \geq K$. Since the integers $n_k$ go to infinity, there is some $k \geq K$ such that $n_k > N$. Thus, there is a term $a_{n_k}$ of the original sequence with index larger than $N$ that is in $U$. This shows that $\{a_n\}$ is frequently in every neighborhood of $p$.

□

**Fact 2.14.** *A cluster point of a subsequence $\{a_{n_k}\}$ is a cluster point of the original sequence $\{a_n\}$.*

## 3. The Extended Real Numbers

The fact that not all subsets of the real line have a sup leads us to introduce the extended real numbers, which are essentially just a notational device.

We define the extended real numbers $\overline{\mathbb{R}}$ by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, where $\infty$ and $-\infty$ are some objects that are not real numbers. We define the ordering on $\overline{\mathbb{R}}$ by $-\infty < r < \infty$ for all $r \in \mathbb{R}$. To indicate the ordering, we can use the interval notation $\overline{\mathbb{R}} = [-\infty, \infty]$.

Every subset of $\overline{\mathbb{R}}$ has a sup. If $A \subseteq \mathbb{R}$ is bounded above, the sup is the same as in the reals. If $A$ is not bounded above, $\sup(A) = \infty$. If $A \subseteq \overline{\mathbb{R}}$ contains $\infty$, then $\infty$ is the only upper bound for $A$, so $\sup(A) = \infty$. Similarly, every subset of $\overline{\mathbb{R}}$ has an inf.

**Fact 3.1.** **[Reflection Trick]** *If* $A \subseteq \overline{\mathbb{R}}$ *is a nonempty set,* $\inf(A) = -\sup(-A)$ *and* $\sup(A) = -\inf(-A)$.

We define addition on $\overline{\mathbb{R}}$ by declaring it to be commutative and defining

$$\begin{gathered} \infty + \infty = \infty \\ r + \infty = \infty, \qquad r \in \mathbb{R}. \\ r - \infty = r + (-\infty) = -\infty, \qquad r \in \mathbb{R} \\ -\infty - \infty = -\infty \\ -\infty + \infty \text{ and } \infty - \infty \text{ are } \textbf{undefined}. \end{gathered}$$

This addition is associative in the following sense: if $a, b, c \in \overline{\mathbb{R}}$ either both of $a+(b+c)$ and $(a+b)+c$ are undefined or both are defined and $a+(b+c) = (a+b)+c$.

**Proposition 3.2.** *Suppose that* $a, A, b, B \in \overline{\mathbb{R}}$ *and* $a \leq A$ *and* $b \leq B$. *Then* $a + b \leq A + B$, *provided both sides are defined.*

*Proof.* As usual with the extended reals, the proof is by considering cases. First, suppose that $A$ and $B$ are real. If $a$ and $b$ are real, we know that $a + b \leq A + B$. The second possibility is that $a + b = -\infty$, in which case $a + b \leq A + B$ is certainly true.

Secondly, $A + B$ might be $\infty$, in which case $a + b \leq A + B$ is certainly true, provided that $a + b$ is defined.

The last possibility is that one of $A$ and $B$ is $-\infty$ and the other is $-\infty$ or real. In this case one of $a$ and $b$ must be $-\infty$ and the other must be $-\infty$ or real. We must have $A + B = -\infty$ and $a + b = -\infty$, so $a + b \leq A + B$ is true. □

We define multiplication by declaring it to be commutative, and using the principal that multiplication by a negative quantity should reverse ordering. Thus, we define

$$\infty \cdot \infty = \infty$$
$$(-\infty) \cdot \infty = -\infty$$
$$(-\infty) \cdot (-\infty) = \infty$$
$$r \cdot \infty = \begin{cases} \infty, & 0 < r < \infty \\ -\infty, & -\infty < r < 0 \end{cases}$$
$$r \cdot (-\infty) = \begin{cases} -\infty, & 0 < r < \infty \\ \infty, & -\infty < r < 0 \end{cases}$$
$$0 \cdot (\pm\infty) = 0.$$

The definition $0 \cdot (\pm\infty) = 0$ (as opposed to leaving it undefined) is usually appropriate in integration theory, but requires caution when dealing with limits, as we will see below. Thus, it's good to keep in mind that caution is required with this definition.

**Exercise 3.3.** The following facts are often used.

(1) Let $A \subseteq \overline{\mathbb{R}}$ be a nonempty set and let $c \in (0, \infty)$. Then $\sup(cA) = c \sup(A)$.
(2) Let $X$ be a set and let $f$ and $g$ be functions $X \to [0, \infty]$. Then
$$\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}.$$
Give an example where strict inequality occurs.

## 4. Sequences in the Extended Real Numbers

We can easily extend the definition of neighborhood to $\overline{\mathbb{R}}$. If $p \in \mathbb{R}$, a set $U \subseteq \overline{\mathbb{R}}$ is a **neighborhood of** $p$ if $U$ contains an interval $(p - \varepsilon, p + \varepsilon)$ for some $\varepsilon > 0$. A set $U$ is a **neighborhood of** $\infty$ if $U$ contains an interval $(\alpha, \infty]$ for some $\alpha \in \mathbb{R}$. Finally, $U$ is a **neighborhood of** $-\infty$ if $U$ contains an interval $[-\infty, \beta)$, for some $\beta \in \mathbb{R}$.

We say that a sequence $\{a_n\}$ in $\overline{\mathbb{R}}$ converges to $p \in \overline{\mathbb{R}}$ if the sequence is eventually in every neighborhood of $p$.

As before, we say that $p$ is a cluster point of $\{a_n\}$ if $\{a_n\}$ is frequently in every neighborhood of $p$.

The following facts are extensions of the facts we know about real sequences to sequences in $\overline{\mathbb{R}}$. The proofs are very similar, with a few, easy, added lines to deal with $\pm\infty$.

**Fact 4.1.** *Let $\{a_n\}$ be a sequence in $\overline{\mathbb{R}}$. If $a_n \to a$, then every subsequence of $\{a_n\}$ converges to $a$.*

**Fact 4.2.** *Let $\{a_n\}$ be a sequence in $\overline{\mathbb{R}}$. Then $p$ is a cluster point of $\{a_n\}$ if and only if there is a subsequence of $\{a_n\}$ that converges to $p$.*

**Fact 4.3.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}}$. Suppose that $a_n \to a$, $b_n \to b$ and that $a_n \leq b_n$ (eventually). Then $a \leq b$.*

*As a corollary, if $c, d \in \overline{\mathbb{R}}$ and $c \leq a_n \leq d$ (eventually), then $c \leq a \leq d$.*

**Fact 4.4.** *Let $\{a_n\}$ be a sequence in $\overline{\mathbb{R}}$. If $a_n$ is increasing, i.e., $a_n \leq a_{n+1}$, then $a_n$ converges to $\sup\{a_n \mid n \in \mathbb{N}\}$. Similarly, if $a_n$ is decreasing, it converges to $\inf\{a_n \mid n \in \mathbb{N}\}$.*

Sums and products of sequences take more care than in the real case.

**Proposition 4.5.** *Let $\{a_n\}$ and $\{b_n\}$ be sequence in $\overline{\mathbb{R}}$ and suppose that $a_n \to a$ and $b_n \to b$. Then, if $a+b$ is defined, $a_n+b_n$ is eventually defined and $a_n+b_n \to a+b$.*

*Proof.* There are 9 cases for the possible values for $a$ and $b$, namely

$$a = \infty, \qquad b = \infty \tag{4.1}$$
$$a = \infty, \qquad b \in \mathbb{R} \tag{4.2}$$
$$a = \infty \qquad b = -\infty \tag{4.3}$$
$$a \in \mathbb{R} \qquad b = \infty \tag{4.4}$$
$$a \in \mathbb{R} \qquad b \in \mathbb{R} \tag{4.5}$$
$$a \in \mathbb{R} \qquad b = -\infty \tag{4.6}$$
$$a = -\infty \qquad b = \infty \tag{4.7}$$
$$a = -\infty, \qquad b \in \mathbb{R} \tag{4.8}$$
$$a = -\infty \qquad b = -\infty \tag{4.9}$$

In cases (4.3) and (4.7), the sum $a + b$ is undefined, so we don't need to consider these cases. We can ignore the case (4.4), since the proof would be the same as in case (4.2), just interchanging the letters $a$ and $b$. Similarly, we can ignore (4.8), since the proof would be the same as in (4.6).

First, consider the case (4.1). Since $(-\infty, \infty]$ is a neighborhood of $\infty$, $a_n$ is eventually not equal to $-\infty$. Similarly, $b_n$ is eventually not equal to $-\infty$. Thus, $a_n + b_n$ is eventually defined. Let $\alpha \in \mathbb{R}$ be arbitrary. Then, $a_n \in (\alpha/2, \infty]$ eventually. Similarly for $b_n$. Thus, for sufficiently large $n$, $a_n+b_n > \alpha/2+\alpha/2 = \alpha$. This shows that $a_n + b_n \to \infty = a + b$.

Next, consider case (4.2). Since $\mathbb{R}$ is a neighborhood of $b$, $b_n$ is eventually real. Thus, $a_n + b_n$ is eventually defined. Since $b_n \to b$, $\{b_n\}$ is bounded in the real numbers, so there is some $M \in \mathbb{R}$ so that $M \leq b_n$ for all sufficiently large $n$. Let $\alpha \in \mathbb{R}$ be arbitrary. Since $a_n \to \infty$, $a_n$ is eventually greater than $\alpha - M$. Thus, for sufficiently large $n$, $a_n + b_n > (\alpha - M) + M = \alpha$. This shows that $a_n + b_n \to \infty = a + b$.

Consider the case (4.5). As in the last case, both sequences are eventually reals, so $a_n + b_n$ is eventually defined. The result then follows from the theorem on the limit of the sum in the real case.

The remaining cases are very similar and are left to the reader. □

The proof of the following proposition about products of sequences is left to the reader.

**Proposition 4.6.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}}$, and suppose that $a_n \to a$ and $b_n \to b$. Then $a_nb_n \to ab$ provided that $ab$ is not of the form $0 \cdot (\pm\infty)$.*

## 5. Lim Sup and Lim Inf

Since this material may be new, I will provide more details than in the review sections.

Let $\{a_n\} \subseteq \overline{\mathbb{R}}$ be a sequence. For each $n \in \mathbb{N}$, define

$$A_n = \{a_k \mid k \geq n\} = \{a_n, a_{n+1}, a_{n+2}, \dots\},$$

the $n$-th tail of the sequence. Clearly, these form a decreasing sequence of sets, i.e., $A_n \supseteq A_{n+1}$. Define

$$A_n^+ = \sup(A_n).$$

Since the $A_n$'s are decreasing, $A_n^+$ is a decreasing sequence of extended reals numbers. Thus, this sequence converges and

$$\lim_{n\to\infty} A_n^+ = \inf_{n\in\mathbb{N}} A_n^+$$

This extended real number is called the **limit superior** of $\{a_n\}$, and is denoted by

$$\limsup_{n\to\infty} a_n \quad \text{or} \quad \overline{\lim_{n\to\infty}}\, a_n.$$

Similarly, we define

$$A_n^- = \inf(A_n).$$

Since the sets $A_n$ are decreasing, $A_n^-$ is an increasing sequence, so

$$\lim_{n\to\infty} A_n^- = \sup_{n\in\mathbb{N}} A_n^-.$$

This extended real number is called the **limit inferior** of $\{a_n\}$ and is denoted by

$$\liminf_{n\to\infty} a_n \quad \text{or} \quad \underline{\lim_{n\to\infty}}\, a_n.$$

We have, of course, $A_n^- \le A_n^+$, so letting $n \to \infty$ gives

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

There is a straight forward version of the reflection trick for lim sup and lim inf. The proof is left as an exercise.

**Proposition 5.1. [Reflection Trick]** *Let $\{a_n\} \subseteq \overline{\mathbb{R}}$ be a sequence. Then,*

(1)
$$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n),$$

(2)
$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

I will generally do the proofs in this section for lim sup and leave the proofs for lim inf as an exercise. The can be done by similar arguments, or by using the reflection trick.

The following is our first basic proposition about lim sup and lim inf.

**Proposition 5.2.** *Let $\{a_n\} \subseteq \overline{\mathbb{R}}$ be a sequence.*

(1) *If*
$$\limsup_{n\to\infty} a_n < \alpha,$$
*then eventually $a_n < \alpha$*

(2) *If, eventually, $a_n \le \alpha$ then*
$$\limsup_{n\to\infty} a_n \le \alpha.$$

(3) *If*
$$\beta < \liminf_{n\to\infty} a_n$$
*then eventually $\beta < a_n$.*

(4) *If, eventually,* $\beta \leq a_n$ *then*

$$\beta \leq \liminf_{n\to\infty} a_n.$$

*Proof.* Consider the first part of the proposition. If

$$\limsup_{n\to\infty} a_n < \alpha,$$

then

$$\inf_{n\in\mathbb{N}} A_n^+ < \alpha,$$

so there is some $N \in \mathbb{N}$ so that $A_N^+ < \alpha$. But $a_n \leq A_N^+$ for $n \geq N$, so $a_n < \alpha$ for $n \geq N$.

For the second part of the Proposition, suppose that $a_n \leq \alpha$ for $n \geq N$. Then, for any $n \geq N$, we have

$$A_n^+ = \sup\{a_n, a_{n+1}, \ldots\} \leq \alpha$$

and so

$$\limsup_{n\to\infty} a_n = \inf_n A_n^+ \leq \alpha.$$

This completes the proof. □

The following Proposition clarifies the meaning of lim sup and lim inf.

**Proposition 5.3.** *Let* $\{a_n\}$ *be a sequence in* $\overline{\mathbb{R}}$. *Then,* $\limsup_n a_n$ *is the largest cluster point of* $\{a_n\}$ *and* $\liminf_n a_n$ *is the smallest cluster point of* $\{a_n\}$, *i.e., if* $p$ *is a cluster point of* $\{a_n\}$, *then*

$$\liminf_{n\to\infty} a_n \leq p \leq \limsup_{n\to\infty} a_n.$$

*Proof.* I'll do the proof for lim sup. We first want to show that $\limsup a_n$ is a cluster point. We distinguish 3 cases.

The first case is $\limsup a_n = \infty$. In this case, we have $\inf_n A_n^+ = \infty$, so we must have $A_n^+ = \infty$ for all $n$. Let $\alpha \in \mathbb{R}$ be given. Given any $N$, we have $\alpha < A_N^+$, so there must be some element of $A_N = \{a_N, a_{N+1}, \ldots\}$ that is bigger than $\alpha$. This shows that the sequence $\{a_n\}$ is frequently in the neighborhood $(\alpha, \infty]$ of $\infty$. Since $\alpha$ was arbitrary, we conclude that $\infty$ is a cluster point of $\{a_n\}$.

For the second case, assume $\limsup a_n = s \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary and let $M \in \mathbb{N}$ be given. Since

$$\inf_n A_n^+ = \limsup_{n\to\infty} a_n = s < s + \varepsilon$$

there is some $N \in \mathbb{N}$ so that $s \leq A_N^+ < s + \varepsilon$. (It follows that $a_k < s + \varepsilon$ for $k \geq N$.) Since $A_n^+$ is a decreasing sequence, we have $s \leq A_n^+ < s + \varepsilon$ for $n \geq N$. Let $K = \max(M, N)$, so we have $s \leq A_K^+ < s + \varepsilon$. But then

$$s - \varepsilon < A_K^+ = \sup\{a_k \mid k \geq K\}.$$

Thus, there must be some $k \geq K$ so that

$$s - \varepsilon < a_k \leq A_K^+ < s + \varepsilon.$$

Thus, there is some index $k \geq M$ so that $a_k$ is in the interval $(s - \varepsilon, s + \varepsilon)$. It follows that the sequence $\{a_n\}$ is frequently in $(s - \varepsilon, s + \varepsilon)$. Since $\varepsilon$ was arbitrary, we conclude that $s$ is a cluster point of $a_n$.

The last case to consider is $\limsup a_n = -\infty$. Let $\beta \in \mathbb{R}$ be arbitrary. Since

$$\inf_n A_n^+ < \beta,$$

there must be some $N$ so that $A_N^+ < \beta$. But for any $n \geq N$ we have

$$a_n \leq A_N^+ < \beta.$$

Thus, the sequence $\{a_n\}$ is eventually in $[-\infty, \beta)$. Since $\beta$ was arbitrary, we conclude that $a_n \to -\infty$, so certainly $-\infty$ is a cluster point.

This completes that proof that $\limsup a_n$ is a cluster point. Note that we have proven that $a_n \to -\infty$ if $\limsup a_n = -\infty$. The corresponding statement for $\liminf$ is that $a_n \to \infty$ if $\liminf a_n = \infty$. These statements will be used later.

To finish the proof, suppose that $p$ is a cluster point of $\{a_n\}$. Our goal is to show $p \leq \limsup a_n$. If $p = -\infty$, there is nothing to do, since $-\infty \leq \limsup a_n$ no matter what $\limsup a_n$ is.

So, suppose that $p > -\infty$. Let $\alpha < p$ be arbitrary. Since $p$ is a cluster point, $a_n$ is frequently greater than $\alpha$ (why?). Thus, for any $n$, the set $\{a_k \mid k \geq n\}$ contains elements that are strictly greater than $\alpha$, so $A_n^+ = \sup\{a_k \mid k \geq n\} > \alpha$. Thus, $\alpha < A_n^+$ for all $n$. Letting $n \to \infty$, we conclude that $\alpha \leq \limsup a_n$. Since $\alpha < p$ was arbitrary, we must have $p \leq \limsup a_n$. This completes the proof. □

An easy application of this Proposition is the following.

**Proposition 5.4.** *Let $\{a_n\} \subseteq \overline{\mathbb{R}}$ be a sequence, and let $\{a_{n_k}\}$ be a subsequence. Then,*

$$\liminf_{n\to\infty} a_n \leq \liminf_{k\to\infty} a_{n_k} \leq \limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

*Proof.* Since $\liminf_{k\to\infty} a_{n_k}$ and $\limsup_{k\to\infty} a_{n_k}$ are cluster points of the subsequence $\{a_{n_k}\}$, they are also cluster points of $\{a_n\}$. Thus, the inequality follows from the last Proposition. □

The next Proposition characterizes convergence in terms of $\limsup$ and $\liminf$.

**Proposition 5.5.** *Let $\{a_n\} \subseteq \overline{\mathbb{R}}$ be a sequence. Then*

$$\lim_{n\to\infty} a_n = a \quad \textit{if and only if} \quad \liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n$$

*Proof.* If $a_n \to a$, then $a$ is the only cluster point of the sequence. Since $\liminf a_n$ and $\limsup a_n$ are cluster points, we must have $a = \liminf a_n = \limsup a_n$.

For the second part of the proof, suppose that

$$\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n.$$

If $a = \infty$, then $\liminf a_n = \infty$. But, as we saw in the proof of Proposition 5.3, this implies $a_n \to \infty = a$. Similarly, if $a = -\infty$, then $\limsup a_n = -\infty$ and we saw in the proof of Proposition 5.3 that this implies $a_n \to -\infty$.

This leaves the case where $a \in \mathbb{R}$. Let $\varepsilon > 0$ be given. Since $\limsup a_n = a < a + \varepsilon$, Proposition 5.2 shows that $a_n$ is eventually less than $a + \varepsilon$. Since $a - \varepsilon < a = \liminf a_n$, Proposition 5.2 shows that $a_n$ is eventually greater than $a - \varepsilon$. Thus, $a_n$ is eventually in the interval $(a - \varepsilon, a + \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we conclude $a_n \to a$. □

To conclude this section, we consider what can be said about the $\limsup$ and $\liminf$ of the sum of two sequences.

**Proposition 5.6.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}}$ and suppose that $a_n + b_n$ is (eventually) defined.*

(1) *We have*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n, \tag{5.1}$$

*provided that the right hand side is defined.*

(2) *If one of the sequences is convergent then equality holds in* (5.1) *(still assuming the right hand side is defined).*

(3) *We have*

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n), \tag{5.2}$$

*provided that the left hand side is defined.*

(4) *If one of the sequences is convergent, equality holds in* (5.2) *(still assuming that the left hand side is defined).*

*Proof.* To prove (5.1), let $c_n = a_n + b_n$ and recall the definitions

$$\begin{aligned}
A_n &= \{a_k \mid k \geq n\} \\
A_n^+ &= \sup(A_n) \\
\limsup_{n\to\infty} a_n &= \lim_{n\to\infty} A_n^+ \\
B_n &= \{b_k \mid k \geq n\} \\
B_n^+ &= \sup(B_n) \\
\limsup_{n\to\infty} b_n &= \lim_{n\to\infty} B_n^+ \\
C_n &= \{c_k \mid k \geq n\} \\
C_n^+ &= \sup(C_n) \\
\limsup_{n\to\infty} c_n &= \lim_{n\to\infty} C_n^+.
\end{aligned}$$

Since we are assuming that the right hand side of (5.1) is defined, $A_n^+ + B_n^+$ is eventually defined, so we can consider $n$'s that are large enough to make this sum defined. For $k \geq n$, we have $a_k \leq A_n^+$ and $b_k \leq B_n^+$. Adding these inequalities gives $c_k = a_k + b_k \leq A_n^+ + B_n^+$. Thus, $A_n^+ + B_n^+$ is an upper bound for $C_n$, so $C_n^+ \leq A_n^+ + B_n^+$. Letting $n$ go to infinity gives (5.1).

For the second part, suppose that $a_n \to a$. Then (5.1) becomes

$$\limsup_{n\to\infty}(a_n + b_n) \leq a + \limsup_{n\to\infty} b_n, \tag{5.3}$$

where we assume that the right hand side is defined. Since $\limsup b_n$ is a cluster point of $\{b_n\}$, there is a subsequence $\{b_{n_k}\}$ so that $b_{n_k} \to \limsup b_n$. Since $a_n \to a$, we also have $a_{n_k} \to a$. Thus,

$$a_{n_k} + b_{n_k} \to a + \limsup_{n\to\infty} b_n. \tag{5.4}$$

However, $\{a_{n_k} + b_{n_k}\}$ is a subsequence of $\{a_n + b_n\}$, so the number on the right of (5.4) is a cluster point of $\{a_n + b_n\}$. Thus, we must have

$$a + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n).$$

by Proposition 5.4. Combining this with (5.3) shows that

$$\limsup_{n\to\infty}(a_n + b_n) = a + \limsup_{n\to\infty} b_n$$

and the proof is complete. $\square$

For an example where inequality holds in (5.1) and (5.2), consider the sequences $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $\limsup a_n = \limsup b_n = 1$, $\liminf a_n = \liminf b_n = -1$ and $\limsup(a_n + b_n) = \liminf(a_n + b_n) = \lim(a_n + b_n) = 0$. Thus, we have

$$\begin{aligned} -2 = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n < \liminf_{n\to\infty}(a_n + b_n) = 0 \\ = \limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = 2. \end{aligned}$$

**Exercise 5.7.** Let $\{a_n\}$ be a sequence in $\overline{\mathbb{R}}$. If $c \in [0, \infty)$, show that

$$\liminf_{n\to\infty}(ca_n) = c \liminf_{n\to\infty} a_n$$
$$\limsup_{n\to\infty}(ca_n) = c \limsup_{n\to\infty} b_n.$$

What happens if $c \in (-\infty, 0)$?

**Exercise 5.8.** If $a_n \leq b_n$ eventually, show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$
$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

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