1 Standard Problems

Problems in this section are in the normal range of difficulty for the course.

Problem 1. [12/11/01]

Recall the definitions from the first problem set on unordered summation.

A. Let X and Y be sets and let $f\colon X \times Y \to [0, \infty]$ be a function. Then

$$\textstyle\sum^{*}_{X\times Y} f = \displaystyle\sum_{x\in X}^{*} \sum_{y\in Y}^{*} f(x,y) = \sum_{y\in Y}^{*} \sum_{x\in X}^{*} f(x,y).$$

B. If $f\colon X \times Y \to \mathbb{R}$ is summable over $X \times Y$, then

$$\textstyle\sum_{X\times Y} f = \displaystyle\sum_{x\in X} \sum_{y\in Y} f(x,y) = \sum_{y\in Y} \sum_{x\in X} f(x,y)$$

Problem 2. [12/11/01]

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable with $m(E) > 0$. Then, for every $\alpha \in (0, 1)$, there is a bounded open interval I such that $\alpha m(I) < m(E \cap I)$ (intuitively, E contains at least the fraction α of I).

Problem 3. [12/11/01]

Recall that a metric space is **separable** if it has a countable dense subset. Show that the space $L^1(\mathbb{R}, \mathcal{L}, m)$ is separable.

Problem 4. [12/11/01]

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let h be a measurable function $X \to [0, \infty]$.

A. For $E \in \mathcal{M}$, define

$$\nu(E) = \int_E h \, d\mu.$$

Show that ν is a measure.

B. If $f\colon X \to [0, \infty]$ is $\mathcal{M}$-measurable, then

$$\int f \, d\nu = \int fh \, d\mu.$$

C. A measurable function $f\colon X \to \mathbb{C}$ is in $L^1(\nu)$ if and only if $fh \in L^1(\mu)$ and, in this case,

$$\int f \, d\nu = \int fh \, d\mu.$$

Problem 5. [12/11/01]

This problem concerns the equivalence of two definitions of the Riemann integral. To recall the definitions briefly, we set up some definitions. Let $I = [a, b]$ be a compact interval. A **partition** P of I is a finite set of points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

These points decompose the interval I into the n subintervals $[x_{i-1}, x_i]$, $i = 1, \dots, n$. We denote the length of the i-th subinterval by $\Delta x_i = x_i - x_{i-1}$. The **mesh** of P is defined by

$$\operatorname{mesh}(P) = \max \{\, \Delta x_i \mid i = 1, \dots, n \,\}$$

Let $f \colon I \to \mathbb{R}$ be a *bounded* function and let $P = \{\, x_0, \dots, x_n \,\}$ be a partition. We define

$$\begin{aligned} M_i &= \sup \{\, f(x) \mid x \in [x_{i-1}, x_i] \,\} \\ m_i &= \inf \{\, f(x) \mid x \in [x_{i-1}, x_i] \,\} \\ U(f; P) &= \sum_{i=1}^{n} M_i \Delta x_i \\ L(f, P) &= \sum_{i=1}^{n} m_i \Delta x_i \\ \overline{\mathrm{I}}_a^b(f) &= \inf \{\, U(f, P) \mid P \text{ a partition of } I \,\} \\ \underline{\mathrm{I}}_a^b(f) &= \sup \{\, L(f; P) \mid P \text{ a partition of } I \,\} \end{aligned}$$

If if the upper integral $\overline{\mathrm{I}}_a^b(f)$ and the lower integral $\underline{\mathrm{I}}_a^b(f)$ are equal, we say f is I-integrable and define the integral $\mathrm{I}_a^b(f)$ to be the common value.

For the second definition, we need some additional machinery. If $P = \{\, x_0, \dots, x_n \,\}$ is a partition, a **choice** ξ **for** P is an indexed collection of points $\{\, \xi_i \,\}_{i=1}^n$ such that $\xi_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. The Riemann sum $S(f; P, \xi)$ is defined to be

$$S(f; P, \xi) = \sum_{i=1}^{n} f(\xi_i) \Delta x_i.$$

We say that f is II-integrable with integral $\mathrm{II}_a^b(f) = J$ if there is a real number J so that

$$\lim_{\operatorname{mesh}(P) \to 0} S(f; P, \xi) = J. \tag{1.1}$$

The definition of this limit is as follows: (1.1) holds if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|S(f; P, \xi) - J| < \varepsilon$$

for every partition P of $[a, b]$ with $\operatorname{mesh}(P) < \delta$ and every choice ξ for P.

Show that these two definitions are equivalent, i.e., f is I-integrable if and only if it is II-integrable and, in this case, $\mathrm{I}_a^b(f) = \mathrm{II}_a^b(f)$.

Problem 6. [12/11/01]

A. **True or false?:** If $f\colon [a, b] \to \mathbb{R}$ is continuous except on a set of measure zero, there is a continuous function g so that $f = g$ a.e. Give a proof or a counterexample.

B. If $f\colon [a, b] \to \mathbb{R}$ is continuous except on a set of measure zero, then f is Lebesgue measurable.

Problem 7. [12/11/01]

This problem works out some of the basic properties of the Fourier Transform in one dimension. These facts can be found in the textbook and many other books, but try it for yourself before peeking.

If $f \in L^1(\mathbb{R}) = L^1(\mathbb{R}, \mathcal{L}, m)$, the function $x \mapsto e^{ix\xi} f(x)$ is L^1 for any $\xi \in \mathbb{R}$ ($i = \sqrt{-1}$), so we can define a function $\hat{f}\colon \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{ix\xi} f(x)\, dx$$

The function $\hat{f}$ is called the **Fourier Transform of** f.

A. If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is bounded and continuous.

B. Prove the **Riemann-Lebesgue Lemma**: If $f \in L^1(\mathbb{R})$, then

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0.$$

Hint: start with the case $f = \chi_{(a,b)}$.

C. Suppose that $f \in L^1(\mathbb{R})$. For $a \in \mathbb{R}$ let $(\tau_a f)(x) = f(x - a)$. Calculate $(\tau_a f)\hat{}$.

D. If $f \in L^1(\mathbb{R})$ and the function $x \mapsto x f(x)$ is L^1, then $\hat{f}$ is differentiable.

E. Let $g(x) = e^{-x^2}$. Compute $\hat{g}$. Hint: find a differential equation satisfied by $\hat{g}$ and solve the differential equation.

Problem 8. [12/11/01]

This problem deals with the translation semigroup on $L^1(\mathbb{R}) = L^1(\mathbb{R}, \mathcal{L}, m)$.

A. If $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, define a function $U_t f$ by

$$(U_t f)(x) = f(x - t).$$

Show that $U_t f \in L^1(\mathbb{R})$ and $\|U_t f\| = \|f\|$. Show that U_t is a linear operator on $L^1(\mathbb{R})$.

B. Show that U_0 is the identity operator and $U_t \circ U_s = U_{t+s}$ for $t, s \in \mathbb{R}$.

C. Fix $f \in L^1(\mathbb{R})$. Show that the function

$$\varphi_f \colon \mathbb{R} \to L^1(\mathbb{R}) \colon t \mapsto U_t f$$

is continuous on all of $\mathbb{R}$ if and only φ_f is continuous at 0.

D. If $f \in L^1(\mathbb{R})$ is continuous with compact support, then φ_f is continuous.

E. For any function $f \in L^1(\mathbb{R})$, φ_f is continuous.

Problem 9. [12/11/01]

Let (X, d) be a metric space. If $A \subseteq X$ is nonempty, and $x \in X$ define $d(x, A)$, the distance from X to A, by

$$d(x, A) = \inf \{ d(x, a) \mid a \in A \} .$$

A. For any $A \neq \emptyset$, the function

$$d(\cdot, A) \colon X \to \mathbb{R} \colon x \mapsto d(x, A)$$

is continuous, indeed, Lipschitz.

B. If A is a nonempty subset of X, then

$$\{ x \in X \mid d(x, A) = 0 \} = \bar{A}$$

(the closure of A).

C. Let A and B be nonempty closed subsets of X. Then there is a continuous function $f \colon X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

D. (optional) Use a computer program to draw the graph of the function f in the last part of the problem when $X = \mathbb{R}$, $A = [0, 1]$ and $B = [3, 4]$.

Problem 10. [12/11/01]

A. Do problem 32, on page 63 of the textbook.

B. Do problem 33, on page 63 of the textbook.

Problem 11. [12/11/01]

Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Let $\mu_0 \colon \mathcal{A} \to [0, \infty]$ be a premeasure and let μ^* be the outer measure generated by μ_0.

A. Show that a set $E \subseteq X$ is μ^*-measurable if and only if

$$\mu_0(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \text{for all } A \in \mathcal{A} \text{ with } \mu_0(A) < \infty$$

B. Suppose that $\mu_0(X) < \infty$. Then $E \subseteq X$ is μ^*-measurable if and only if

$$\mu^*(E) + \mu^*(E^c) = \mu_0(X).$$

Problem 12. [12/11/01]

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \colon X \to [0, \infty]$ be a measurable function. Then, there is a sequence of measurable sets $\{ E_k \}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{E_k}.$$

Hint: $E_1 = \{ x \mid f(x) \geq 1 \}$, then define the E_k's inductively.

Problem 13. [12/12/01] Fix a number $h > 0$. If $B \subseteq \mathbb{R}^{n-1}$, define $B_h \subseteq \mathbb{R}^n$ to be the set of all points $(x, h) = (x_1, x_2, \dots, x_{n-1}, h)$ where $x = (x_1, x_2, \dots, x_{n-1}) \in B$. For $B \subseteq \mathbb{R}^{n-1}$ and $h > 0$ define

$$\Gamma(B, h) = \{ ty \mid t \in [0, 1],\ y \in B_h \},$$

which we can think of as a generalized cone in $\mathbb{R}^n$ with vertex at the origin, base B_h and height h.

1. Let $B \subseteq \mathbb{R}^n$ be Lebesgue measurable. Assuming that $\Gamma(B, h)$ is Lebesgue measurable in $\mathbb{R}^n$, show that

$$m^n(\Gamma(B, h)) = \frac{h}{n} m^{n-1}(B).$$

2. If $B \subseteq \mathbb{R}^{n-1}$ is a Borel set, $\Gamma(B, h)$ is a Borel set.
3. If $B \subseteq \mathbb{R}^{n-1}$ is Lebesgue measurable, $\Gamma(B, h)$ is Lebesgue measurable. Hint: use the formula.

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2 Somewhat more Challenging Problems

Problems in this section are somewhat more challenging.

Problem 1. [12/11/01]
Let X be an uncountable set and let ν be the counting measure on X. Show that $L^1(\nu)$ is **not** separable.

Problem 2. [12/11/01]
Show that for each $t \in \mathbb{R}$,

$$\int_0^\infty e^{-x^2} \cos(2xt)\, dx = \frac{\sqrt{\pi}}{2} e^{-t^2}.$$

Hints: Let $F(t)$ be the integral above. Find, and solve, a differential equation for F.

Problem 3. [12/11/01]
Recall the gamma function, defined on $H^+ = \{\, z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \,\}$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Prove that Γ is infinitely differentiable on H^+ (in the sense of the derivative of a function of a complex variable) and find integral formulas for the derivatives.

If you insist, you can avoid the derivative of a function of a complex variable by restricting z to $(0, \infty)$.

Problem 4. [12/11/01]
Show that

$$F(t) = \int_0^{\pi/2} \ln(t\cos(x))\, dx$$

makes sense as a Lebesgue integral for all $t > 0$. Show that

$$F(t) = \frac{\pi}{2} \ln(t/2), \qquad t > 0.$$

Hints: Find F' and then find F. Evaluate the constant of integration by using

$$\int_0^{\pi/2} \ln(\cos(x)), dx = \int_0^{\pi/2} \ln(\sin(x))\, dx.$$

Problem 5. [12/11/01]

1. The functions $f(x) = \cos(x^2)$ and $g(x) = \sin(x^2)$ are **not** Lebesgue integrable on $[0, \infty)$.

2. Show that for all $r > 0$,

$$\int_0^\infty \left[\int_0^r e^{-xy^2} \sin(x)\, dx \right] dy = \int_0^r \left[\int_0^\infty e^{-xy^2} \sin(x)\, dy \right] dx.$$

   By letting $r \to \infty$, show that

$$\int_0^\infty \frac{\sin(x)}{\sqrt{x}}\, dx = \frac{\sqrt{2\pi}}{2}.$$

3. Evaluate the Fresnel integrals

$$\int_0^\infty \sin(x^2)\, dx, \qquad \int_0^\infty \cos(x^2)\, dx$$

   as (conditionally convergent) improper Riemann integrals.

Problem Set

Practice Problems for the
First Semester of Real Analysis

Math 5322, Fall 2001

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Good luck!