A NOTE ON THE LEBESGUE COVERING LEMMA

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The purpose of this note is to give a proof of the Lebesgue Covering Lemma, which we used in class.

First, we recall a couple of definitions. Let (X, d) be a metric space and let $E \subseteq X$ be a nonempty set. We define $\operatorname{diam}(E)$, the **diameter of** E by

$$\operatorname{diam}(E) = \sup\{d(x, y) \mid x, y \in E\},$$

(this might be ∞, of course). The diameter of the empty set is 0.

Let $\mathcal{U}$ be an open cover of X, i.e., $\mathcal{U}$ is a collection of open subsets of X and every point $x \in X$ is in some element U of $\mathcal{U}$. A number $\lambda > 0$ is a **Lebesgue Number for** $\mathcal{U}$ if every set of diameter less than λ is contained is some element in the cover $\mathcal{U}$, i.e., if $\operatorname{diam}(E) < \lambda$, there is some $U \in \mathcal{U}$ such that $E \subseteq U$.

Lebesgue Covering Lemma. If X is a compact metric space, every open cover of X has a Lebesgue number.

The proof will occupy the remainder of this note.

Let $\mathcal{U}$ be an open cover of our compact metric space X. For every point $x \in X$, select an element $U(x)$ of $\mathcal{U}$ so that $x \in U(x)$. Since $U(x)$ is open, we can find some radius $r(x) > 0$ such that $B_{r(x)}(x) \subseteq U(x)$.

The collection of open balls

$$\mathcal{B} = \{B_{r(x)/2}(x) \mid x \in X\}$$

is clearly an open cover of X. Since X is compact, there is a finite subcover, say

$$\mathcal{B}' = \{B_{r(x_i)/2}(x_i) \mid i = 1, 2, \ldots, n\}.$$

Let

$$\lambda = \min\{r(x_i)/2 \mid i = 1, 2, \ldots, n\},$$

so $\lambda > 0$.

Of course we want to show that λ is a Lebesgue number for $\mathcal{U}$. To do this, suppose that $E \subseteq X$ and $\operatorname{diam}(E) < \lambda$.

Fix a point $p \in E$. The point p must be in some element of the cover $\mathcal{B}'$, say

$$p \in B_{r(x_i)/2}(x_i).$$

Let q be any point in E. Then

$$d(p, q) \leq \operatorname{diam}(E) < \lambda \leq r(x_i)/2.$$

Then

$$\begin{aligned} d(x_i, q) &\leq d(x_i, p) + d(p, q) \\ &< r(x_i)/2 + r(x_i)/2 = r(x_i). \end{aligned}$$

This shows that

$$E \subseteq B_{r(x_i)}(x_i).$$

But then

$$E \subseteq B_{r(x_i)}(x_i) \subseteq U(x_i).$$

so E is contained in the element $U(x_i)$ of $\mathcal{U}$. This completes the proof.

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