Problem Set

Problems on Unordered Summation

Math 5323, Fall 2001

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ANSWERS

1 Unordered Sums of Real Terms

In calculus and real analysis, one defines the convergence of an infinite series

$$\sum_{n=1}^{\infty} a_n \tag{1.1}$$

of reals numbers as follows: For each postive integer n, define

$$S_n = \sum_{k=1}^{n} a_k,$$

the n-th partial sum of the series. If the sequence $\{ S_n \}$ converges to a real number S, the series converges and the sum is S.

This definition depends on the particular order of the terms a_n—if we reorder the terms, we get a different sequence of partial sums, and so possibly a different result.

In this problem set, we want to consider how to add up infinitly many terms (including, perhaps, more than countably many terms) in a way that does not depend on the ordering.

In our series (1.1) the sequence $\{ a_n \}$ of terms amounts to a function from the index set $\mathbb{N} = \{ 1, 2, 3, \dots \}$ to the real numbers $\mathbb{R}$. In general we will consider an arbitary index set X, and a function $f\colon X \to \mathbb{R}$.

If $F \subseteq X$ is a finite set, then the finite sum

$$\sum_{x \in F} f(x),$$

makes sense, since the order of the terms does not matter in a finite sum. For brevity, we write

$$\textstyle\sum_F f = \displaystyle\sum_F f = \sum_{x \in F} f(x).$$

The emptyset $\emptyset \subseteq X$ is a finite set with $\sum_\emptyset f = 0$, for any function f.

Note that if $F_1, F_2 \subseteq X$ are finite and $F_1 \cap F_2 = \emptyset$, then

$$\textstyle\sum_{F_1 \cup F_2} f = \sum_F f + \sum_{F_2} f,$$

since there are no duplicate terms in the sum on the left.

Since we'll be using finite subsets of X a lot, we introduce the notation

$$\operatorname{Fin}(X) = \{ F \subseteq X \mid F \text{ is finite} \}.$$

In the definition of the sum of a series like (1.1), the idea is the we can add up finitely many terms, and if we add in more and more terms we get closer and closer to the sum—but we have a particular order in which to put in additional terms.

In the following definition, we use the idea that adding in more terms should get closer to the sum, but we don't impose any order on which terms to take next.

Definition 1.1. *Let X be a nonempty set and let $f\colon X \to \mathbb{R}$. We say that f is* ***summable over*** *X if there is a real number S with the following property: For every $\varepsilon > 0$, there is a finite set $F_0 \subseteq X$ such that*

$$\left|\sum\nolimits_F f - S\right| < \varepsilon, \qquad \textit{whenever } F \supseteq F_0 \textit{ and } F \textit{ is finite.}$$

Lemma 1.2. *There is at most one number S that satisfies the property in Definition 1.1.*

Proof. Suppose that S_1 and S_2 have the property in the defintion. Let $\varepsilon > 0$ be arbitary. Then there is an $F_1 \in \mathrm{Fin}(X)$ such that

$$F \in \mathrm{Fin}(X),\ F \supseteq F_1 \implies \left|\sum\nolimits_F f - S_1\right| < \varepsilon.$$

Similarily, there is an $F_2 \in \mathrm{Fin}(X)$ such that

$$F \in \mathrm{Fin}(X),\ F \supseteq F_2 \implies \left|\sum\nolimits_F f - S_3\right| < \varepsilon.$$

If we set $F = F_1 \cup F_2 \in \mathrm{Fin}(X)$, then $F \supseteq F_1, F_2$ and we have

$$\begin{aligned} |S_1 - S_2| &= \left|S_1 - \sum\nolimits_F f + \sum\nolimits_F f - S_2\right| \\ &\le \left|S_1 - \sum\nolimits_F f\right| + \left|\sum\nolimits_F f - S_2\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Thus, we have

$$|S_1 - S_2| < 2\varepsilon.$$

Since ε was arbitary, we must have $|S_1 - S_2| = 0$, i.e., $S_1 = S_2$ □

If f is summable, we will denote the unique number S in the defintion by

$$\sum\nolimits_X f \qquad \text{or} \qquad \sum_X f \qquad \text{or} \qquad \sum_{x \in X} f(x),$$

and we introduce the notation

$$\mathrm{S}(X) = \{\, f\colon X \to \mathbb{R} \mid f \text{ is summable}\,\}.$$

Problem 1. Suppose that $f\colon X \to \mathbb{R}$.

A. If X itself is finite, we've given two definitions of the symbol $\sum_X f$. Show these two definitions are the same.

B. If $f(x) = 0$ for all $x \in X$, then $\sum_X f = 0$.

Problem 2. The space $\mathrm{S}(X)$ of summable functions is a vector space (under the pointwise operations) and the mapping $\mathrm{S}(X) \to \mathbb{R}\colon f \mapsto \sum_X f$ is linear. In other words,

A. If $f, g \in \mathrm{S}(X)$, then $f + g$ is summable and

$$\sum_X (f+g) = \sum_X f + \sum_X g. \tag{1.2}$$

Answer:

We want to show that the number $\sum_X f + \sum_X g$ satisfies the condition in Definition 1.1.

Let $\varepsilon > 0$ be arbitrary. Since f is summable, there is a finite set F_1 such that

$$F \in \mathrm{Fin}(X),\ F \supseteq F_1 \implies \left|\sum_F f - \sum_X f\right| < \varepsilon.$$

Similarly, there is a finite set F_2 such that

$$F \in \mathrm{Fin}(X),\ F \supseteq F_2 \implies \left|\sum_F g - \sum_X g\right| < \varepsilon.$$

Let $F_3 = F_1 \cup F_2$, so F_3 is finite and $F_3 \supseteq F_1, F_2$. Thus, if $F \supseteq F_3$, we have

$$\begin{aligned}\left|\sum_F (f+g) - \left[\sum_X f + \sum_X g\right]\right| &= \left|\sum_F f - \sum_X f + \sum_F g - \sum_X g\right| \\ &\le \left|\sum_F f - \sum_X f\right| + \left|\sum_F g - \sum_X g\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, this completes the proof.

B. If $f \in \mathrm{S}(X)$ and $c \in \mathbb{R}$, cf is summable and

$$\sum_X cf = c \sum_X f. \tag{1.3}$$

Answer:

We need to show the number on the right of (1.3) satisfies the condition in Definition 1.1.

Let $\varepsilon > 0$ be arbitrary. Since f is summable, there is a finite set F_0 such that

$$F \in \mathrm{Fin}(X),\ F \supseteq F_0 \implies \left|\sum_F f - \sum_X f\right| < \varepsilon.$$

Then, if $F \supseteq F_0$, we have

$$\begin{aligned}\left|\sum\nolimits_F cf - \sum\nolimits_X cf\right| &= |c|\left|\sum\nolimits_F f - \sum\nolimits_X f\right| \\ &\leq |c|\varepsilon.\end{aligned}$$

Since ε was arbitrary, $|c|\varepsilon$ can be as small as we like, so this completes the proof.

Problem 3. Let $f\colon X \to \mathbb{R}$.

A. The function f is summable over X if and only if the following **Cauchy Condition** is satisfied: For every $\varepsilon > 0$, the is a finite set F_0 so that

$$F, F' \in \operatorname{Fin}(X),\ F, F' \supseteq F_0 \implies \left|\sum\nolimits_F f - \sum\nolimits_{F'} f\right| < \varepsilon.$$

Answer:

First suppose that f is summable. We show that the Cauchy condition holds. Let $\varepsilon > 0$ be given. Then there is a finite set F_0 such that

$$F \in \operatorname{Fin}(X),\ F \supseteq F_0 \implies \left|\sum\nolimits_F f - \sum\nolimits_X f\right| < \varepsilon.$$

Suppose that F, F' are finite sets that contain F_0. Then,

$$\begin{aligned}\left|\sum\nolimits_F f - \sum\nolimits_{F'} f\right| &= \left|\sum\nolimits_F f - \sum\nolimits_X f + \sum\nolimits_X f - \sum\nolimits_{F'} f\right| \\ &\leq \left|\sum\nolimits_F f - \sum\nolimits_X f\right| + \left|\sum\nolimits_X f - \sum\nolimits_{F'} f\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, the Cauchy Condition holds.

For the second part of the proof, suppose the Cauchy condition holds. We must show that f is summable.

Applying the Cauchy condition with $\varepsilon = 1/n$ for $n \in \mathbb{N}$ gives us a sequence of finite sets F_n so that

$$\text{(A)} \qquad F, F' \in \operatorname{Fin}(X),\ F, F \supseteq F_n \implies \left|\sum\nolimits_F f - \sum\nolimits_{F'} f\right| < \frac{1}{n}.$$

If we set $E_n = F_1 \cup F_2 \cup \cdots \cup F_n$, then $E_1 \subseteq E_2 \subseteq E_3 \cdots$ and $E_n \supseteq F_n$.

We claim that the sequence $\left\{ \sum_{E_n} f \right\}$ is a Cauchy sequence of real numbers. To see this, let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that $1/N < \varepsilon$. If $m, n \geq N$ then $E_m, E_n \supseteq E_N$, so by (A), we have $\left|\sum_{E_n} f - \sum_{E_m} f\right| < 1/N < \varepsilon$. Thus, $\left\{ \sum_{E_n} f \right\}$ is Cauchy.

Since a Cauchy sequence of real numbers converges, there is a number S so that $\sum_{E_n} f \to S$ as $n \to \infty$.

We claim that $S = \sum_X f$. To see this, let $\varepsilon > 0$ be arbitrary, and choose $N_1 \in \mathbb{N}$ so that $1/N_1 < \varepsilon$. Then, as above,

$$\text{(B)} \qquad F, F' \in \operatorname{Fin}(X),\ F, F' \supseteq E_{N_1} \implies \left|\sum\nolimits_F f - \sum\nolimits_{F'} f\right| < \varepsilon.$$

On the other hand, we can find N_2 so that

$$\text{(C)} \qquad n \geq N_2 \implies \left|\sum\nolimits_{E_n} -S\right| < \varepsilon.$$

Let $N = \max(N_1, N_2)$. Suppose that $F \supseteq E_N$. Then we have $\left|\sum_F f - \sum_{E_N} f\right| < \varepsilon$, by (B) and we have $\left|\sum_{E_N} f - S\right| < \varepsilon$, by (C). Thus,

$$\begin{aligned}\left|\sum\nolimits_F f - S\right| &= \left|\sum\nolimits_F f - \sum\nolimits_{E_N} f + \sum\nolimits_{E_N} f - S\right| \\ &\leq \left|\sum\nolimits_F f - \sum\nolimits_{E_N} f\right| + \left|\sum\nolimits_{E_N} f - S\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, we conclude that f is summable.

B. The Cauchy Condition is equivalent to the following **Alternate Cauchy Condition**: For all $\varepsilon > 0$ there is a finite set G such that

$$F \in \operatorname{Fin}(X),\ F \cap G = \emptyset \implies \left|\sum\nolimits_F f\right| < \varepsilon.$$

Answer:

First suppose the Cauchy condition holds. Let $\varepsilon > 0$ be arbitrary. Then there is a finite set F_0 so that

$$F, F' \in \operatorname{Fin}(X),\ F, F' \supseteq F_0 \implies \left|\sum\nolimits_F f - \sum\nolimits_{F'} f\right| < \varepsilon.$$

Suppose that $H \in \operatorname{Fin}(X)$ and that $H \cap F_0 = \emptyset$. Then both F_0 and $H \cup F_0$ contain F_0, so we have

$$\begin{aligned}\left|\sum\nolimits_H f\right| &= \left|\sum\nolimits_H f + \sum\nolimits_{F_0} f - \sum\nolimits_{F_0} f\right| \\ &= \left|\sum\nolimits_{H\cup F_0} f - \sum\nolimits_{F_0} f\right| < \varepsilon.\end{aligned}$$

Thus, the Alternate Cauchy Condition holds.

For the second part of the proof, assume that Alternate Cauchy Condition holds, and we want to show the Cauchy Condition holds. To do this, let $\varepsilon > 0$ be given and let G be as in the Alternate Cauchy Condition. Suppose that F is finite and $F \supseteq G$. Then we can write $F = G \cup H$, where $H \cap G = \emptyset$, so $|\sum_H f| < \varepsilon$.

Now suppose that $F_1, F_2 \supseteq G$ and write $F_i = G \cup H_i$ $(i = 1, 2)$ as above. Then we have

$$\begin{aligned}
\left|\sum_{F_1} f - \sum_{F_2} f\right| &= \left|\sum_{G \cup H_1} f - \sum_{G \cup H_2} f\right| \\
&= \left|\sum_{G} f + \sum_{H_1} f - \sum_{G} f - \sum_{H_2} f\right| \\
&= \left|\sum_{H_1} f - \sum_{H_2} f\right| \\
&\leq \left|\sum_{H_1} f\right| + \left|\sum_{H_2} f\right| \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Since ε was arbitrary, we conclude the Cauchy Condition holds.

If $A \subseteq X$, we can consider the function $f|_A \colon A \to \mathbb{R} \colon a \mapsto f(a)$, the restriction of f to A. If $f|_A$ is summable over A, we say f is summable over A and write the sum as just $\sum_A f$ (instead of $\sum_A f|_A$).

Problem 4. Let $f \colon X \to \mathbb{R}$ and let A be a nonempty subset of X. If f is summable over X, then f is summable over A.

Answer:
We will use the Cauchy Condition to show that f is summable over A.

To show the Cauchy Condition holds on A, let $\varepsilon > 0$ be arbitrary. Let $B = X \setminus A$, so X is the disjoint union of A and B. Since f is summable over X, is satisfies the Cauchy Condition on X, so there is a set $G \in \operatorname{Fin}(X)$ so that

$$\text{(A)} \qquad F_1, F_2 \in \operatorname{Fin}(X),\ F_1, F_2 \supseteq G \implies \left|\sum_{F_1} f - \sum_{F_2} f\right| < \varepsilon.$$

The set G is the disjoint union of $G \cap A$ and $G \cap B$.

Suppose that $H_1, H_2 \in \operatorname{Fin}(A)$ and that $H_1, H_2 \supseteq G \cap A$. Then $H_1 \cup (G \cap B) \supseteq G$ and $H_2 \cup (G \cap B) \supseteq G$. Thus, we have

$$\left|\sum_{H_1 \cup (G \cap B)} f - \sum_{H_2 \cup (G \cap B)} f\right| < \varepsilon,$$

by (A). But

$$\begin{aligned}
\left|\sum_{H_1 \cup (G \cap B)} f - \sum_{H_2 \cup (G \cap B)} f\right| &= \left|\sum_{H_1} f + \sum_{G \cap B} f - \sum_{H_2} f - \sum_{G \cap B} f\right| \\
&= \left|\sum_{H_1} f - \sum_{H_2} f\right|
\end{aligned}$$

and so we have

$$\left|\sum_{H_1} f - \sum_{H_2} f\right| < \varepsilon.$$

We conclude that f satisfies the Cauchy Condition on A, and so is summable on A.

Problem 5. Let $f\colon X \to \mathbb{R}$ and suppose that X is the disjoint union of two sets A and B. Then f is summable over X if and only if f is summable over both A and B and, in this case,

$$\sum_X f = \sum_A f + \sum_B f. \tag{1.4}$$

Answer:
First suppose that f is summable over both A and B. We want to show that f is summable over X and that (1.4) holds.

Let $\varepsilon > 0$ be arbitrary. Since f is summable over A, there is a set $F_A \in \mathrm{Fin}(A)$ so that

$$F \in \mathrm{Fin}(A),\ F \supseteq F_A \implies \left|\sum_F f - \sum_A f\right| < \varepsilon.$$

Similarly, there is a set $F_B \in \mathrm{Fin}(B)$ so that

$$F \in \mathrm{Fin}(B),\ F \supseteq F_B \implies \left|\sum_F f - \sum_B f\right| < \varepsilon.$$

Set $F_X = F_A \cup F_B$. If $F \in \mathrm{Fin}(X)$ and $F \supseteq F_X$, we can write F and the disjoint union of $F \cap A$ and $F \cap B$, with $F \cap A \supseteq F_A$ and $F \cap B \supseteq F_B$. Then we have

$$\begin{aligned}
\left|\sum_F f - \left[\sum_A f + \sum_B f\right]\right| &= \left|\sum_{(F\cap A)\cup(F\cap B)} f - \sum_A f - \sum_B f\right| \\
&= \left|\sum_{F\cap A} f + \sum_{F\cap B} f - \sum_A f - \sum_B f\right| \\
&\le \left|\sum_{F\cap A} f - \sum_A f\right| + \left|\sum_{F\cap B} f - \sum_B f\right| \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

We conclude that f is summable over X and (1.4) hold.

For the second part of the proof, we assume that f is summable over X. But then, by the previous problem, f is summable over A and B and hence (1.4) holds by the first part of the proof.

The result of the last problem is easily extended inductively to the case where X is partitioned into a finite number of subsets $A_1, A_2 \ldots, A_n$. In this case f is summable over X if and only if it is summable over each of the A_i and, if this is so,

$$\sum_X f = \sum_{A_1} f + \sum_{A_2} f + \cdots + \sum_{A_n} f.$$

A simple, but useful, corollary to the last problem is the following.

Corollary 1.3. *Suppose that $A \subseteq X$ and that $f\colon X \to \mathbb{R}$ is zero outside A. Then f is summable over X if and only if it is summable over A and, in this case,*

$$\sum_X f = \sum_A f$$

Problem 6. Let $f\colon X \to \mathbb{R}$ be summable. Then, for every $\varepsilon > 0$, there is a finite set F_0 such that

$$X \supseteq A \supseteq F_0 \implies \left|\sum_A f - \sum_X f\right| < \varepsilon.$$

This differs from the definition of summablitity because A is allowed to be infinite.

Answer:
Let $\varepsilon > 0$ be given. Since f is summable, there is a finite set F_0 such that

$$\text{(A)} \qquad \left|\sum_F f - \sum_X f\right| < \varepsilon$$

for all finite $F \supseteq F_0$.

Suppose that A is any subset of X that contains F_0. Since f is summable over X, it is summable over A, so there is a finite set G_0 such that

$$\text{(B)} \qquad \left|\sum_F f - \sum_A f\right| < \varepsilon$$

for all finite sets F with $A \supseteq F \supseteq G_0$.

Let $F = F_0 \cup G_0$. Then $F \subseteq A$ and $F \supseteq F_0, G_0$. Thus, both (A) and (B) hold. Then we have

$$\begin{aligned}\left|\sum_A f - \sum_X f\right| &= \left|\sum_A f - \sum_F f + \sum_F f - \sum_X f\right| \\ &\leq \left|\sum_A f - \sum_F f\right| + \left|\sum_F f - \sum_X f\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

This completes the proof.

Our next step is to extend one direction of Problem ??? to arbitary partitions of X. Recall that a **partition** $\mathcal{P}$ of X is a collection of nonempty subsets of X such that the elements of $\mathcal{P}$ are pairwise disjoint and

$$\bigcup \mathcal{P} = \bigcup_{P \in \mathcal{P}} P = X.$$

It follows that every point of X is in exactly one of the elements of P

Problem 7. Let $f \in \mathrm{S}(X)$ and let $\mathcal{P}$ be any partition of X. Since f is summable over every subset of X, we can define a function $g \colon \mathcal{P} \to \mathbb{R} \colon P \mapsto g(P)$ by

$$g(P) = \sum\nolimits_P f,$$

Then, the function g is summable over $\mathcal{P}$ and

$$\sum\nolimits_{\mathcal{P}} g = \sum\nolimits_X f.$$

Answer:
Let $\varepsilon > 0$ be given. By the last problem, there an $F_0 \in \mathrm{Fin}(X)$ so that

$$\text{(A)} \qquad X \supseteq A \supseteq F_0 \implies \left| \sum\nolimits_A f - \sum\nolimits_X f \right| < \varepsilon.$$

Every point $x \in X$ is contained in exactly one element of $\mathcal{P}$, call it $P(x)$. Define $\mathcal{F}_0$ by

$$\mathcal{F}_0 = \{ \, P(x) \mid x \in F_0 \, \} \, .$$

Then $\mathcal{F}_0$ is a finite subcollection of $\mathcal{P}$ and certainly $\bigcup \mathcal{F}_0$ (the union of the elements of $\mathcal{F}$) contains F_0. Suppose that $\mathcal{F}$ is a finite subcollection of $\mathcal{P}$ that contains $\mathcal{F}_0$. Then $\bigcup \mathcal{F} \supseteq \bigcup \mathcal{F}_0 \supseteq F_0$.

If we label the elements of $\mathcal{F}$ by $\mathcal{F} = \{ \, P_1, P_2, \ldots, P_k \, \}$, we have

$$\begin{aligned}
\sum\nolimits_{\mathcal{F}} g = \sum_{P \in \mathcal{F}} g & \\
&= g(P_1) + g(P_2) + \cdots + g(P_k) \\
&= \sum\nolimits_{P_1} f + \sum\nolimits_{P_2} f + \cdots + \sum\nolimits_{P_k} f \\
&= \sum_{P_1 \cup \cdots \cup P_k} f \\
&= \sum\nolimits_{\bigcup \mathcal{F}} f.
\end{aligned}$$

But, (A) implies that

$$\left| \sum\nolimits_{\bigcup \mathcal{F}} f - \sum\nolimits_X f \right| < \varepsilon,$$

because $\bigcup \mathcal{F} \supseteq F_0$. Thus, we have

$$\left| \sum\nolimits_{\mathcal{F}} g - \sum\nolimits_X f \right| < \varepsilon.$$

for all finite subcollections $\mathcal{F}$ that contain $\mathcal{F}_0$. This completes the proof.

Note that the converse of this theorem is false. That is, it is possible that f is summable over every element of $\mathcal{P}$, so that g is defined, and that g is summable over $\mathcal{P}$, but f is not summable over X.

For an example like this, let $X = \mathbb{N}$ and define $f\colon \mathbb{N} \to \mathbb{R}$ by $f(n) = (-1)^n$. Let $\mathcal{P}$ be the partition

$$\mathcal{P} = \{ \{ 1, 2 \}, \{ 3, 4 \}, \{ 5, 6 \}, \dots \}.$$

Then f is summable over each element of $\mathcal{P}$ and g is summable over $\mathcal{P}$, but f is not summable over $\mathbb{N}$. To see this, note that if f was summable over $\mathbb{N}$, it would be summable over every subset of $\mathbb{N}$. In particular, it would be summable over $\{ 2, 4, 6, 8, \dots \}$, which is not the case (why?).

2 Unordered Sums of Postive Terms

Suppose that $f\colon X \to [0, \infty]$. If $F \in \operatorname{Fin}(X)$, the finite sum $\sum_F f$ makes sense (it is ∞, if f takes the value ∞ at some point of F). For functions with non-negative values, the finite sums have two monotonicity properties. First,

$$F_1, F_2 \in \operatorname{Fin}(X),\ F_1 \subseteq F_2 \implies \sum\nolimits_{F_1} f \leq \sum\nolimits_{F_2} f.$$

Secondly, if $g\colon X \to [0, \infty]$ is another function, we have

$$f \leq g,\ F \in \operatorname{Fin}(X) \implies \sum\nolimits_F f \leq \sum\nolimits_F g.$$

For any function $f\colon X \to [0, \infty]$, we define

$$\sum\nolimits_X^* f = \sum_X{}^* f = \sum_{x \in X}{}^* f(x)$$

by

$$\sum\nolimits_X^* f = \sup \left\{ \sum\nolimits_F f \mid F \in \operatorname{Fin} X \right\},$$

which may be ∞.

Problem 8. Suppose that $f, g\colon X \to \mathbb{R}$.

A. If A and B are subsets of X,

$$A \subseteq B \implies \sum\nolimits_A^* f \leq \sum\nolimits_B^* f \tag{2.1}$$

Answer:

By definition,

$$\sum\nolimits_A^* f = \sup\left\{ \sum\nolimits_F f \mid F \subseteq A,\ F \text{ finite} \right\} \tag{A}$$

$$\sum\nolimits_B^* f = \sup\left\{ \sum\nolimits_F f \mid F \subseteq B,\ F \text{ finite} \right\}. \tag{B}$$

The set on the right-hand side of (B) is larger than the set in (A), and so has a larger sup. This establishes (2.1).

B. For any subset A of X,

$$f \leq g \implies \sum\nolimits_A^* f \leq \sum\nolimits_A^* g. \tag{2.2}$$

Answer:

Suppose $f \leq g$. For any finite set $F \subseteq A$, we have

$$\sum\nolimits_F f \leq \sum\nolimits_F g \leq \sum\nolimits_A^* g.$$

Thus, $\sum\nolimits_A^* g$ is an upper bound for the set

$$\left\{ \sum\nolimits_F f \mid F \subseteq A,\ F \text{ finite} \right\},$$

and hence is bigger than or equal to the sup of this set, which is $\sum\nolimits_A^* f$. Thus, (2.2) holds.

Problem 9.

In this problem, we establish that $\sum\nolimits_X^*$ is a close to linear as it can be.

A. If $f, g \colon X \to [0, \infty]$ then

$$\sum\nolimits_X^* (f+g) = \sum\nolimits_X^* f + \sum\nolimits_X^* g.$$

Answer:
If F is a finite set, then

$$\sum_F (f+g) = \sum_F f + \sum_F f \le \sum_X^* f + \sum_X^* g.$$

Taking the sup over F, we conclude that

$$\sum_X^* (f+g) \le \sum_X^* f + \sum_X^* g.$$

To complete the proof, it will suffice to show the opposite inequality

$$\sum_X^* f + \sum_X^* g \le \sum_X^* (f+g) \tag{A}$$

If $\sum_X^* (f+g) = \infty$, this inequality is true, so we are reduced to considering the case $\sum_X^* ((f+g)) < \infty$. Since $f, g \le f+g$ we have

$$\sum_X^* f, \sum_X^* g \le \sum_X^* (f+g) < \infty.$$

Let $\varepsilon > 0$ be arbitrary. Then there is a finite set F_1 so that

$$\sum_X^* f - \varepsilon < \sum_{F_1} f.$$

Similarly, there is a finite set F_2 so that

$$\sum_X^* g - \varepsilon < \sum_{F_2} g.$$

Let $F = F_1 \cup F_2$. Then we have

$$\sum_X^* f - \varepsilon < \sum_{F_1} f \le \sum_F f \tag{2.3}$$

$$\sum_X^* g - \varepsilon < \sum_{F_2} g \le \sum_F g \tag{2.4}$$

Adding these inequalities (ignoring the middle term) gives

$$\sum_X^* f + \sum_X^* g - 2\varepsilon < \sum_F f + \sum_F g = \sum_F (f+g) \le \sum_X^* (f+g).$$

Thus, we have

$$\sum_X^* f + \sum_X^* g - 2\varepsilon < \sum_X^* (f+g).$$

Since ε was arbitrary, this implies (A)

B. If $c \in [0, \infty)$ and $f\colon X \to [0, \infty]$, then

$$\sum\nolimits_X^* cf = c \sum\nolimits_X^* f,$$

where we're using the definition $0 \cdot \infty = 0$.

Answer:

If $c = 0$, both sides of the equation in the problem are zero. So consider the case $c > 0$. For finite F, we have $\sum_F cf = c \sum_F f$ (which is the distributive law). Thus, we have

$$\left\{ \sum\nolimits_F cf \mid F \in \operatorname{Fin} X \right\} = \left\{ c \sum\nolimits_F f \mid F \in \operatorname{Fin}(X) \right\} = c \left\{ \sum\nolimits_F f \mid F \in \operatorname{Fin}(X) \right\},$$

from which the equation in the problem follows by an elementary property of sup's.

Problem 10.

A. Suppose that $A \subseteq X$ and that $f\colon X \to [0, \infty]$ is zero outside of A. Then

$$\sum\nolimits_A^* f = \sum\nolimits_X^* f.$$

Answer:

Since $A \subseteq X$, we have

$$\sum\nolimits_A^* f \leq \sum\nolimits_X^* f,$$

so it will suffice to establish the opposite inequality. Suppose that $F \subseteq X$ is finite. Since $f = 0$ outside A, we have $\sum_F f = \sum_{F \cap A} f$. Thus,

$$\sum\nolimits_F f = \sum\nolimits_{F \cap A} f \leq \sum\nolimits_A^* f.$$

Sup'ing over F yields

$$\sum\nolimits_X^* f \leq \sum\nolimits_A^* f$$

and the proof is complete.

B. Suppose that $f\colon X \to [0, \infty]$. Let A be a subset of X and define χ_A, **the characteristic function of** A, by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Then, we have

$$\sum\nolimits_A^* f = \sum\nolimits_X^* f\chi_A.$$

Answer:

Since $f\chi_A$ is zero outside A, $\sum\nolimits_X^* f\chi_A = \sum\nolimits_A^* f\chi_A$. But, $f = f\chi_A$ on A, so $\sum\nolimits_A^* f\chi_A = \sum\nolimits_A^* f$.

C. Let $f\colon X \to [0,\infty]$ and suppose that X is the disjoint union of A and B. Then

$$\sum\nolimits_X^* f = \sum\nolimits_A^* f + \sum\nolimits_B^* f.$$

This result is easily extended to a partition of X into finitely many subsets.

Answer:

Since X is the disjoint union of A and B, $\chi_A + \chi_B = 1$. Thus, $f = f\chi_A + f\chi_B$ and so $\sum\nolimits_X^* f = \sum\nolimits_X^* f\chi_A + \sum\nolimits_X^* f\chi_B$. Now apply the previous part to the problem.

Problem 11. Let $f\colon X \to [0,\infty]$ and let $\mathcal{P}$ be a partition of X. Define $g\colon \mathcal{P} \to [0,\infty]$ by

$$g(P) = \sum\nolimits_P^* f.$$

Then

$$\sum\nolimits_{\mathcal{P}}^* g = \sum\nolimits_X^* f.$$

Answer:
If $\mathcal{F} = \{ P_1, P_2, \dots P_k \}$ is a finite subcollection of $\mathcal{P}$, then

$$\begin{aligned}\sum\nolimits_{\mathcal{F}}^* g &= g(P_1) + g(P_2) + \dots + g(P_k)\\ &= \sum\nolimits_{P_1}^* f + \sum\nolimits_{P_2}^* f + \dots \sum\nolimits_{P_k}^* f\\ &= \sum_{P_1\cup P_2\cup\dots\cup P_k}^* f\\ &= \sum\nolimits_{\bigcup\mathcal{F}}^* f.\end{aligned}$$

Thus, if $\mathcal{F}$ is a finite subcollection of $\mathcal{P}$,

$$\sum\nolimits_{\mathcal{F}}^* g = \sum\nolimits_{\bigcup\mathcal{F}}^* f \le \sum\nolimits_X^* f.$$

Taking the sup over finite subcollection $\mathcal{F} \subseteq \mathcal{P}$, we conclude that

$$\sum\nolimits_{\mathcal{P}}^{*} g \leq \sum\nolimits_{X}^{*} f.$$

To get the opposite inequality, let F be a finite subset of X and set

$$\mathcal{F} = \{ P(x) \mid x \in F \}$$

where $P(x)$ is the unique element of $\mathcal{P}$ containing x. Then $\mathcal{F}$ is a finite subcollection of $\mathcal{P}$ and $\bigcup \mathcal{F} \supseteq F$. Thus,

$$\sum\nolimits_{F} f \leq \sum\nolimits_{\bigcup \mathcal{F}}^{*} f = \sum\nolimits_{\mathcal{F}}^{*} g \leq \sum\nolimits_{\mathcal{P}}^{*} g.$$

Supping over F gives

$$\sum\nolimits_{X}^{*} f \leq \sum\nolimits_{\mathcal{P}}^{*} g,$$

and the proof is complete.

Problem 12. Let $f\colon X \to [0, \infty]$ and suppose that X is countably infinite. Let $\{ x_n \}_{n=1}^{\infty}$ be **any** enumeration of X. Then

$$\sum\nolimits_{X}^{*} f = \sum_{n=1}^{\infty} f(x_n), \tag{2.5}$$

where the right-hand side is the limit of the partial sums, which may be infinity.

Answer:
Let $F_n = \{ x_1, x_2, \ldots, x_n \}$. Then,

$$s_n = \sum_{k=1}^{n} f(x_k),$$

the nth partial sum of the series on the right of (2.5) is the same as $\sum_{F_n} f$. Thus,

$$s_n = \sum\nolimits_{F_n} f \leq \sum\nolimits_{X}^{*} f.$$

Since the partial sums s_n converge upward to the sum of the series s, we conclude that $s \leq \sum\nolimits_{X}^{*} f$.

For the opposite inequality, suppose that $F \subseteq X$ is finite. Find the maximum value of n such that $x_n \in F$. Say this maximum value is N. Then $F \subseteq F_N$, so

$$\sum\nolimits_{F} f \leq \sum\nolimits_{F_N} f = s_N \leq s.$$

Supping over F gives $\sum\nolimits_{X}^{*} f \leq s$ and the proof is complete.

If $f\colon X \to [0,\infty]$, we define **the carrier of** f, denoted $\operatorname{carr}(f)$, by

$$\operatorname{carr}(f) = \{\, x \in X \mid f(x) \neq 0 \,\}\,.$$

From our previous results, we have

$$\sum\nolimits_X^* f = \sum\nolimits_{\operatorname{carr}(f)}^* f.$$

If $\operatorname{carr}(f)$ is finite, the right hand side of this equation would be a finite sum. If $\operatorname{carr}(f)$ is countably infinite, we can apply the last problem.

Problem 13. Suppose that $f\colon X \to [0,\infty]$ and that

$$\sum\nolimits_X^* f < \infty.$$

Then $\operatorname{carr}(f)$ is countable.

Answer:
Let $E_n = \{\, x \in X \mid f(x) \geq 1/n \,\}$ for $n \in \mathbb{N}$. Then

$$\operatorname{carr}(f) = \bigcup_{n=1}^{\infty} E_n$$

(To see this, note is $f(x) > 0$ then $f(x) \geq 1/n$ for some n.)

We claim that each of the sets E_n is finite. To prove this, suppose, for a contradiction, that E_N is infinite. Then, for any $k \in \mathbb{N}$, we can find a subset F_k of E_N consisting of k elements. We then have

$$\frac{k}{N} \leq \sum\nolimits_{F_k} f \leq \sum\nolimits_X^* f$$

Thus, $k/N \leq \sum\nolimits_X^* f$. Since k is arbitrary, this would imply $\sum\nolimits_X^* f = \infty$, contrary to our hypothesis.

We conclude that each E_n is finite. Then $\operatorname{carr}(f)$ is a countable union of countable sets, and hence countable.
