Problem Set

Problem Set #2

Math 5322, Fall 2001

December 3, 2001

ANSWERS

Problem 1. [Problem 18, page 32]

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

a. For any $E \subseteq X$ and $\varepsilon > 0$ the exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

Answer:

By definition,

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(A_i) \mid \{ A_i \}_{i=1}^{\infty} \subseteq \mathcal{A},\ E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Thus, given $\varepsilon > 0$, we can find some sequence $\{ A_i \}_{i=1}^{\infty} \subseteq \mathcal{A}$ so that

$$E \subseteq \bigcup_{i=1}^{\infty} A_i \quad \text{and} \quad \sum_{i=1}^{\infty} \mu_0(A_i) < \mu^*(E) + \varepsilon.$$

Define a set $A \supseteq E$ by

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Since A is a countable union of sets in $\mathcal{A}$, $A \in \mathcal{A}_\sigma$. Since μ^* is subadditive,

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu_0(A_i) < \mu^*(E) + \varepsilon,$$

since $\mu^* = \mu_0$ on $\mathcal{A}$. This completes the proof.

b. If $\mu^*(E) < \infty$ then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

Answer:

Since the σ-algebra $\mathcal{M}$ of μ^*-measurable sets contains $\mathcal{A}$ and is closed under countable unions and intersections, $\mathcal{A}_{\sigma\delta} \subseteq \mathcal{M}$.

For the first part of the proof, assume that $E \in \mathcal{M}$ and $\mu^*(E) < \infty$. For each $n \in \mathbb{N}$, we can apply the previous part of the problem to find a set $A_n \in \mathcal{A}_\sigma$ such that $E \subseteq A_n$ and $\mu^*(A_n) < \mu^*(E) + 1/n$. Define B by

$$B = \bigcap_{n=1}^{\infty} A_n.$$

Then $E \subseteq B$ and $B \in \mathcal{A}_{\sigma\delta}$, since it is a countable intersection of sets in $\mathcal{A}_\sigma$. For every n, we have

$$\mu^*(E) \le \mu^*(B) \le \mu^*(A_n) < \mu^*(E) + 1/n.$$

Since n is arbitrary, we must have $\mu^*(B) = \mu^*(E)$.

Since $E \subseteq B$, B is the disjoint union of E and $B \setminus E$. These three sets are in $\mathcal{M}$, so we can use the additivity of μ^* on $\mathcal{M}$ to get

$$\mu^*(B) = \mu^*(E) + \mu^*(B \setminus E).$$

All three terms are finite and $\mu^*(E) = \mu^*(B)$, so simple algebra shows $\mu^*(B \setminus E) = 0$.

For the second part of the proof, suppose that $E \subseteq X$ and there is a set $B \in \mathcal{A}_{\sigma\delta}$ so that $E \subseteq B$ and $\mu^*(B \setminus E) = 0$. As remarked above, $\mathcal{A}_{\sigma\delta} \subseteq \mathcal{M}$ so B is measurable. We showed in class that if $F \subseteq X$ and $\mu^*(F) = 0$, then F is μ^*-measurable. Thus, in the present situation, $B \setminus E$ is measurable. But $E = B \setminus (B \setminus E)$ (check!). Since $\mathcal{M}$ is closed under taking set differences, we conclude that E is measurable. This completes the proof.

c. If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Answer:

First suppose that E is μ^*-measurable and $\mu^*(E) = \infty$. Since μ_0 is σ-finite we can find disjoint sets $F_j \in \mathcal{A}$, $j \in \mathbb{N}$ with $\mu_0(F_j) < \infty$ and $X = \bigcup_{j=1}^\infty F_j$. Set $E_j = E \cap F_j$. Then the sets E_j are disjoint, $\mu^*(E_j) \le \mu^*(F_j) = \mu_0(F_j) < \infty$ and $E = \bigcup_j E_j$.

As a first try, we might invoke the previous part of the problem to find sets $B_j \supseteq E_j$ with $B_j \in \mathcal{A}_{\sigma\delta}$ and $\mu^*(B_j \setminus E_j) = 0$. The next step would be to try to define our desired set B as $\bigcup_j B_j$. Unfortunately, this *won't* work. Since $\mathcal{A}_{\sigma\delta}$ is defined to be the collection of countable *intersections* of sets in $\mathcal{A}_\sigma$, there's no obvious reason why $\mathcal{A}_{\sigma\delta}$ should be closed under taking countable *unions*. Thus, we don't know that $\bigcup_j B_j$ will be in $\mathcal{A}_{\sigma\delta}$.

Note, however that $\mathcal{A}_\sigma$ is closed under countable unions. If $\{ A_i \}_{i=1}^\infty \subseteq \mathcal{A}_\sigma$, then for each i there is a sequence $\{ A_i^k \}_{k=1}^\infty \subseteq \mathcal{A}$ such that

$$A_i = \bigcup_{k=1}^\infty A_i^k.$$

But then if $A = \bigcup_i A_i$, we have

$$A = \bigcup_{i=1}^\infty \left[\bigcup_{k=1}^\infty A_i^k \right] = \bigcup_{(i,k) \in \mathbb{N} \times \mathbb{N}} A_i^k,$$

a countable union of elements of $\mathcal{A}$ (check!). Thus, $A \in \mathcal{A}_\sigma$.

So, we proceed as follows. For each $n \in \mathbb{N}$, we invoke part (a) to get a set $A_j^n \in \mathcal{A}_\sigma$ such that $E_j \subseteq A_j^n$ and

$$\mu^*(A_j^n) < \mu^*(E_j) + \frac{1}{n2^j}.$$

Since A_j^n and E_j are measurable and have finite measure, this gives us

$$\mu^*(A_j^n \setminus E_j) < \frac{1}{n2^j}.$$

We then define

$$A^n = \bigcup_{j=1}^{\infty} A_j^n.$$

As remarked above, $A^n \in \mathcal{A}_\sigma$. Next, we claim that

$$(*) \qquad A^n \setminus E \subseteq \bigcup_{j=1}^{\infty} (A_j^n \setminus E_j).$$

To see this, suppose that $x \in A^n \setminus E$. Since $x \in A^n$, there is some j such that $x \in A_j^n$. Since $x \notin E$ and $E_j \subseteq E$, $x \notin E_j$. Thus $x \in A_j^n \setminus E_j$. This completes the proof of the claim.

From $(*)$, we have

$$\mu^*(A^n \setminus E) \leq \sum_{j=1}^{\infty} \mu^*(A_j^n \setminus E_j) < \sum_{j=1}^{\infty} \frac{1}{n2^j} = \frac{1}{n}.$$

Now define

$$B = \bigcap_{n=1}^{\infty} A^n.$$

Then, $B \in \mathcal{A}_{\sigma\delta}$ and $E \subseteq B$. For every n we have

$$\mu^*(B \setminus E) \leq \mu^*(A^n \setminus E) < \frac{1}{n},$$

and so $\mu^*(B \setminus E) = 0$, which completes the first part of the proof.

The proof of the converse implication is the same as in part (b).

Problem 2. [Problem 25, page 39]

Complete the proof of Theorem 1.19.

Thus, we want to prove that the following conditions on a set $E \subseteq \mathbb{R}$ are equivalent *a.* $E \in \mathcal{M}_\mu$

b. $E = V \setminus N_1$, where V is a G_δ set and $\mu(N_1) = 0$.

c. $E = H \cup N_2$, where H is an F_σ set and $\mu(N_2) = 0$.

Here μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$ and $\mathcal{M}_\mu$ is its domain (the μ^*-measurable sets, where μ^* is the outer measure used in the construction of μ).

Answer:
The proof of the implication (a) $\implies$ (b) is pretty similar to what we did in the last problem. In general, a countable union of G_δ sets is not a G_δ set. But, we can start with Proposition 1.18.

So, suppose that $E \in \mathcal{M}_\mu$. Let $\{ F_j \}$ be a sequence of disjoint measurable sets such that $\mathbb{R} = \bigcup_j F_j$ and $\mu(F_j) < \infty$. We can take the F_j's to be bounded intervals, for example. Let $E_j = E \cap F_j$, so the E_j's are disjoint measurable sets whose union is E and each E_j has finite measure.

By Proposition 1.18, for each $n \in \mathbb{N}$ and j we can find an open set U_j^n such that $E_j \subseteq U_j^n$ and

$$\mu(U_j^n) < \mu(E_j) + \frac{1}{n2^j}.$$

Since $U_j^n = E_j \cup (U_j^n \setminus E_j)$ (disjoint union), we have

$$\mu(U_j^n) = \mu(E_j) + \mu(U_j^n \setminus E_j).$$

All the terms in this equation are finite, so we can rearrange the equation to get

$$\mu(U_j^n \setminus E_j) = \mu(U_j^n) - \mu(E_j) < \frac{1}{n2^j}.$$

Define a set U^n by $U^n = \bigcup_j U_j^n$. Then $E \subseteq U^n$ and U^n is open, since any union of open sets is open. We have

$$U^n \setminus E \subseteq \bigcup_{j=1}^{\infty} (U_j^n \setminus E_j)$$

(check), and so

$$\mu(U^n \setminus E) \leq \sum_{j=1}^{\infty} \mu(U_j^n \setminus E_j) < \sum_{j=1}^{\infty} \frac{1}{n2^j} = \frac{1}{n}.$$

Now we define

$$V = \bigcap_{n=1}^{\infty} U^n,$$

so V is a G_δ set and $V \supseteq E$. For any n, we have

$$\mu(V \setminus E) \leq \mu(U^n \setminus E) < \frac{1}{n},$$

so we must have $\mu(V \setminus E) = 0$. We have

$$E = V \setminus (V \setminus E) = V \setminus N_1$$

where V is a G_δ set and $N_1 = V \setminus E$ is a nullset. This completes the proof that (a) $\implies$ (b).

We next prove that (a) $\implies$ (c). We'll give a direct proof, which is easier than the proof of (a) $\implies$ (b). So suppose that $E \in \mathcal{M}_\mu$, let $\{ F_j \}$ be a partition of $\mathbb{R}$ into sets of finite measure and let $E_j = E \cap F_j$.

Fix j for the moment. For each $n \in \mathbb{N}$ we can apply Proposition 1.18 to get a compact set $K_n \subseteq E_j$ such that

$$\mu(E_j) - \frac{1}{n} < \mu(K_n)$$

and so

$$\mu(E_j \setminus K_n) < \frac{1}{n}.$$

Set $H_j = \bigcup_n K_n$. Since each K_n is closed, H_j is an F_σ set and $H_j \subseteq E_j$. For each n,

$$E_j \setminus H_j \subseteq E_j \setminus K_n$$

so

$$\mu(E_j \setminus H_j) \leq \mu(E_j \setminus K_n) < \frac{1}{n}.$$

Thus, we have $\mu(E_j \setminus H_j) = 0$, where $H_j \subseteq E_j$ is an F_σ set. Thus, we have $E_j = H_j \cup N_j$ where $N_j = E_j \setminus H_j$ is a nullset.

Taking the union over j, we have

$$E = \left(\bigcup_{j=1}^{\infty} H_j \right) \cup \left(\bigcup_{j=1}^{\infty} N_j \right).$$

Since F_σ is closed under countable unions (by an argument similar to the last problem), the first set on the right of this equation is an F_σ set. Of course a countable union of nullsets is a nullset, so the second set on the right of the last equation is a nullset. This completes the proof of (a) $\implies$ (c).

Next, we prove that (b) $\implies$ (a). This is pretty easy. Suppose that $E = V \setminus N_1$ where V is a G_δ set and N_1 is a nullset. Of course a nullset is measurable. We know that $\mathcal{M}_\mu$ contains the Borel sets, and hence the G_δ sets. Since the σ-algebra $\mathcal{M}_\mu$ is closed under complements and finite intersections, we conclude that $E \in \mathcal{M}_\mu$.

The proof that (c) $\implies$ (a) is similar. Suppose $E = H \cup N_2$ where H is an F_σ set and N_2 is a nullset. Then $N_2 \in \mathcal{M}_\mu$ and $H \in \mathcal{M}_\mu$ since $\mathcal{M}_\mu$ contains the Borel sets and hence the F_σ sets. Since $\mathcal{M}_\mu$ is closed under finite unions, we conclude that $E \in \mathcal{M}_\mu$.

This completes the proof.

Several people observed that once you know (a) $\implies$ (c), you can use this and de Morgan's laws to prove (a) $\implies$ (b) (or vice-versa). Perhaps the simplest proof of our Proposition would be to prove (a) $\implies$ (c) as above, then use de Morgan's laws to get (a) $\implies$ (b), and then do the reverse implications as above.

Problem 3. **[Problem 26, page 39]**

Prove Proposition 1.20 (Use Theorem 1.18.)

Thus, we want to prove that if $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is a finite [disjoint] union of open intervals such that $\mu(E \triangle A) < \varepsilon$

Here μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$. The symbol $\triangle$ denotes the symmetric difference of the sets, i.e., $E \triangle A = (E \setminus A) \cup (A \setminus E)$.

Answer:
Suppose $E \in \mathcal{M}_\mu$ with $\mu(E) < \infty$ and let $\varepsilon > 0$ be given.

By Theorem 1.18 we can find an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ and $\mu(U) < \mu(E) + \varepsilon$. Since U and E have finite measure $\mu(U \setminus E) = \mu(U) - \mu(E) < \varepsilon$.

Since U is an open subset of $\mathbb{R}$, U can be written as a countable disjoint union of open intervals. Suppose first that the number of intervals is infinite, say

$$U = \bigcup_{k=1}^{\infty} I_k,$$

where each I_k is an open interval. We have

$$\sum_{k=1}^{\infty} \mu(I_k) = \mu(U) < \infty.$$

Since this series converges, there is some n such that

$$(*) \qquad \sum_{k=n+1}^{\infty} \mu(I_k) < \varepsilon.$$

We define A by

$$A = \bigcup_{k=1}^{n} I_k.$$

If there are only finitely many intervals in U, we can label them as $I_1, I_2, \dots, I_n$ and let $I_k = \emptyset$ for $k \geq n+1$. Then we define $A = U$ and $(*)$ still holds.

We now have $E \setminus A \subseteq U \setminus A$ so

$$\begin{aligned}\mu(E \setminus A) &\leq \mu(U \setminus A)\\ &= \mu(U) - \mu(A)\\ &= \sum_{k=1}^{\infty} \mu(I_k) - \sum_{k=1}^{n} \mu(I_k)\\ &= \sum_{k=n+1}^{\infty} \mu(I_k) < \varepsilon.\end{aligned}$$

On the other hand, $A \setminus E \subseteq U \setminus E$, so

$$\mu(A \setminus E) \leq \mu(U \setminus E) < \varepsilon.$$

Thus,

$$\mu(E \bigtriangleup A) \leq \mu(E \setminus A) + \mu(A \setminus E) < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete.

Problem 4. [Problem 28, page 39]

Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$ and $\mu_F((a,b)) = F(b-) - F(a)$.

Answer:
Recall that μ_F is constructed so that on the algebra of h-intervals it takes the values

$$\mu_F((a,b]) = F(b) - F(a),$$

and (hence) that μ_F is finite on bounded subsets of $\mathbb{R}$.

To calculate $\mu_F(\{a\})$, note that

$$\text{(A)} \qquad \{a\} = \bigcap_{n=1}^{\infty} (a - 1/n, a].$$

Certainly a is in the right-hand side and any number bigger that a is not. If $x < a$ then $x < a - 1/n$ for sufficiently large n, so x is not in the intersection on the right-hand side of (A).

The intervals $(a - 1/n, a]$ form a decreasing sequence of sets, so by continuity from above (Theorem 1.8d) we have

$$\mu_F(\{a\}) = \lim_{n\to\infty} \mu_F((a - 1/n, a]) = \lim_{n\to\infty} [F(a) - F(a - 1/n)].$$

Since F is increasing,

$$\lim_{n\to\infty} F(a-1/n) = \lim_{x\to a^-} F(x) = \sup\{\, F(x) \mid x < a \,\} = F(a-).$$

Thus, we have

$$\mu_F(\{\, a \,\}) = F(a) - F(a-). \tag{B}$$

Next, consider $\mu_F([a,b])$. We have $[a,b] = \{\, a \,\} \cup (a,b]$ (disjoint union), so

$$\begin{aligned} \mu_F([a,b]) &= \mu_F(\{\, a \,\}) + \mu_F((a,b]) \\ &= F(a) - F(a-) + F(b) - F(a) \\ &= F(b) - F(a-) \end{aligned}$$

so

$$\mu_F([a,b]) = F(b) - F(a-) \tag{C}$$

Next, consider $\mu_F((a,b))$, where we have to allow a and/or b to be infinity. If b is infinity, then (a,∞) is an h-interval and the definition of μ_F gives

$$\mu_F((a,\infty)) = F(\infty) - F(a).$$

The proposed formula

$$\mu_F((a,b)) = F(b-) - F(a) \tag{D}$$

is correct because $F(\infty) = \lim_{x\to\infty} F(x)$. If $a = -\infty$ and $b = \infty$ then $(a,b) = (-\infty,\infty)$ is an h-interval and the definition of μ_F gives

$$\mu_F((-\infty,\infty)) = F(\infty) - F(-\infty),$$

which fits into formula (D) again.

If both a and b are finite, we have $(a,b) = (a,b] \setminus \{\, b \,\}$, so

$$\begin{aligned} \mu_F((a,b)) &= \mu_F((a,b]) - \mu_F(\{\, b \,\}) \\ &= F(b) - F(a) - [F(b) - F(b-)] \\ &= F(b-) - F(a), \end{aligned}$$

so (D) holds.

Finally, consider intervals of the form $[a,b)$, where b might be infinity. We can write $[a,b) = \{\, a \,\} \cup (a,b)$ (disjoint union) so

$$\begin{aligned} \mu_F([a,b)) &= \mu_F(\{\, a \,\}) + \mu_F(a,b) \\ &= F(a) - F(a-) + F(b-) - F(a) \\ &= F(b-) - F(a-), \end{aligned}$$

Note that this is valid if $F(b-) = F(\infty) = \infty$.

Problem 5. **[Problem 29, page 39]**

Let E be a Lebesgue measurable set.

a. If $E \subseteq N$, where N is the nonmeasurable set described in Section 1.1, then $m(E) = 0$.

Answer:

Recall the construction in Section 1.1. We define an equivalence relation $\sim$ on $[0,1)$ by $x \sim y$ if $x - y \in \mathbb{Q}$. We let N be a set that contains exactly one element from each equivalence class (using the Axiom of Choice). Set $R = \mathbb{Q} \cap [0,1)$. For any set $S \subseteq [0,1)$ and r in R we define

$$S_r = \{ x + r \mid x \in S \cap [0, 1-r) \} \cup \{ x + r - 1 \mid x \in S \cap [1-r, 1) \} .$$

We can then argue that if S is measurable, $m(S_r) = m(S)$, using the translation invariance of Lebesgue measure. We also argued that

$$[0,1) = \bigcup_{r \in R} N_r, \qquad \text{(disjoint union)}.$$

If N was measurable, we would have

$$1 = m([0,1)) = \sum_{n=1}^{\infty} m(N),$$

which is impossible since the right-hand side can only be 0 or ∞.

For the first part of the present problem, we want to show that if $E \subseteq N$ is Lebesgue measurable, then $m(E) = 0$. Well, for each $r \in R$ we have $E_r \subseteq N_r$, as defined above. Since the sets N_r are pairwise disjoint,

$$F = \bigcup_{r \in R} E_r$$

is a countable disjoint union of measurable sets contained in $[0,1)$. Thus, we have

$$1 \geq m(F) = \sum_{r \in R} m(E_r) = \sum_{i=1}^{\infty} m(E).$$

If $m(E) > 0$, the sum on the right would be ∞, so we must have $m(E) = 0$.

b. If $m(E) > 0$, then E contains a non-measurable set. (It suffices to assume that $E \subseteq [0,1)$. In the notation of Section 1.1, $E = \bigcup_{r \in R} E \cap N_r$.)

Answer:

Briefly, if $E \subseteq \mathbb{R}$ and $m(E) > 0$, we can write

$$E = \bigcup_{n=-\infty}^{\infty} E \cap [n, n+1), \qquad \text{(disjoint union)}.$$

Since $m(E) > 0$, at least one of the sets $E \cap [n, n+1)$ must have nonzero measure. Select one such set $F = E \cap [n, n+1)$. If F contains a nonmeasurable set, so does E. The set $F - n$ is contained in $[0, 1)$ and has the same measure as F. If we prove that every subset of $[0, 1)$ of nonzero measure contains a nonmeasurable set, then $F - n$ will contain a non-measurable set A. But then $A + n \subseteq F$ is nonmeasurable (if $A + n$ was measurable, so would be $(A + n) - n = A$).

Thus, it will suffice to consider the case where $E \subseteq [0, 1)$ and $m(E) > 0$.

First, observe that the process we've defined above that sends S to S_r is invertible. Indeed, $S_0 = S$ and for $t \in (0, 1)$ we define $\varphi_t \colon [0, 1) \to [0, 1)$ by

$$\varphi_t(x) = \begin{cases} x + t, & x \in [0, 1-t) \\ x + t - 1, & x \in [1-t, 1). \end{cases}$$

Then $S_t = \varphi_t(S)$. It is easy to check that $\varphi_t([0, 1-t)) = [t, 1)$ and $\varphi_t([1-t, 1)) = [0, t)$. It is then easy to check that the inverse of φ_t is φ_{1-t}. Thus

$$(S_t)_{1-t} = S.$$

From the first part of the problem, we can conclude that if F is measurable and $F \subseteq N_r$ for some r then $m(F) = 0$. To see this, note that if $F \subseteq N_r$ then $F_{1-r} \subseteq N$. As we've argued before, F_{1-r} is measurable and $m(F_{1-r}) = m(F)$. From the first part of the problem, $m(F_{1-r}) = 0$, so we conclude that $m(F) = 0$.

Now suppose that $E \subseteq [0, 1)$ and $m(E) > 0$. The interval $[0, 1)$ is the disjoint union of the sets N_r for $r \in R$, so E is the disjoint union of the sets $E \cap N_r$. If all of the sets $E \cap N_r$ were measurable, we would have

$$m(E) = \sum_{r \in R} m(E \cap N_r) = \sum_{r \in R} 0 = 0,$$

which contradicts the assumption that $m(E) > 0$. Thus at least one of the sets $E \cap N_r$ must be nonmeasurable and we conclude that E contains a nonmeasurable set.

Before going into the problems on Page 48ff. of the book, lets recall some of the definitions involved.

Let $(X, \mathcal{M})$ be a measurable space, so $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ-algebra. If $A \subseteq X$, we can define a σ-algebra $\mathcal{M}|_A \subseteq \mathcal{P}(A)$ by

$$\mathcal{M}|_A = \{\, E \cap A \mid E \in \mathcal{M} \,\}. \tag{0.1}$$

If A itself is in $\mathcal{M}$, we have

$$\mathcal{M}|_A = \{\, E \mid E \subseteq A,\ E \in \mathcal{M} \,\}.$$

If $(Y, \mathcal{N})$ is a measurable space and $f\colon X \to Y$, we say that f **is measurable on** $A \subseteq X$ if $f|_A\colon A \to Y$ (the restriction of f to A) is measurable for the σ-algebra $\mathcal{M}|_A$, which means that $(f|_A)^{-1}(N) = f^{-1}(N) \cap A \in \mathcal{M}|_A$ for all $N \in \mathcal{N}$. If $A \in \mathcal{M}$, then f is measurable on A iff $f^{-1}(N) \cap A \in \mathcal{M}$ for all $N \in \mathcal{N}$. Clearly, if f is a measurable function $X \to Y$, then f is measurable on every subset of A of X since, in this case, $f^{-1}(N) \cap A \in \mathcal{M}|_A$ for all $\mathcal{N}$, because $f^{-1}(N) \in \mathcal{M}$.

The following proposition is pretty easy, but useful.

Proposition 0.1. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and let $f\colon X \to Y$ be a function. Let $\{ E_\alpha \}_{\alpha \in A}$ be a countable (finite or infinite) collection of measurable subsets of X such that*

$$X = \bigcup_{\alpha \in A} E_\alpha.$$

Then, f is measurable if and only if f is measurable on each E_α, i.e., f is measurable if and only if for each $N \in \mathcal{N}$, $f^{-1}(N) \cap E_\alpha \in \mathcal{M}$ for all $\alpha \in A$.

Proof. If f is measurable, then $f^{-1}(N) \in \mathcal{M}$ for all $N \in \mathcal{N}$. Since each $E_\alpha \in \mathcal{M}$, we have $f^{-1}(N) \cap E_\alpha \in \mathcal{M}$.

Conversely, suppose that for each $N \in \mathcal{N}$, $f^{-1}(N) \cap E_\alpha$ is measurable for all α. Then

$$f^{-1}(N) = \bigcup_{\alpha \in A} f^{-1}(N) \cap E_\alpha$$

is a countable union of sets in $\mathcal{M}$, so $f^{-1}(N) \in \mathcal{M}$. Thus, f is measurable. □

Problem 6. [Problem 1, page 48]

Let $(X, \mathcal{M})$ be a measurable space. Let $f\colon X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{ -\infty \}) \in \mathcal{M}$, $f^{-1}(\{ \infty \}) \in \mathcal{M}$ and f is measurable on Y.

Answer:
Suppose first that f is measurable. Then f is measurable on Y, as discussed above. The sets $\{ -\infty \}$ and $\{ \infty \}$ are closed sets, hence Borel sets, in $\overline{\mathbb{R}}$, so $f^{-1}(\{ -\infty \})$ and $f^{-1}(\{ \infty \})$ are measurable.

For the second part of proof suppose $f^{-1}(\{ -\infty \})$ and $f^{-1}(\{ \infty \})$ are measurable and f is measurable on Y. The sets $f^{-1}(\{ -\infty \})$, $f^{-1}(\{ \infty \})$ and Y form a partition of X. Since $f^{-1}(\{ -\infty \})$ and $f^{-1}(\{ \infty \})$ are measurable, $Y = X \setminus (f^{-1}(\{ -\infty \}) \cup f^{-1}(\{ \infty \}))$ is measurable. To show f is measurable it will suffice to show it is measurable on the three sets in our partition. Of course, f is measurable on Y by assumption. On the set $f^{-1}(\{ -\infty \})$, f is constant and a constant function is always measurable. Similarly, f is constant, and hence measurable, on $f^{-1}(\{ \infty \})$. This completes the proof.

Problem 7. [Problem 2, page 48]
Let $(X, \mathcal{M})$ be a measurable space. Suppose that $f, g \colon X \to \overline{\mathbb{R}}$ are measurable.

a. fg is measurable (where $0 \cdot (\pm\infty) = 0$).

Answer:

We'll consider two solutions, in slightly different spirits.

First Solution. Since f is measurable, we can partition X into the measurable sets $F_1 = f^{-1}(-\infty)$, $F_2 = f^{-1}((-\infty, 0))$, $F_3 = f^{-1}(0)$, $F_4 = f^{-1}((0, \infty))$ and $F_5 = f^{-1}(\infty)$. ($f^{-1}(a)$ is the same thing as $f^{-1}(\{ a \})$.) We have a similar partition $G_1 = g^{-1}(-\infty)$, $G_2 = g^{-1}((-\infty, 0))$, $G_3 = g^{-1}(0)$, $G_4 = g^{-1}((0, \infty))$, $G_5 = g^{-1}(\infty)$ for g. Taking pairwise intersections, we get a partition of X into the 25 measurable sets (don't panic!) $F_i \cap G_j$. To show fg is measurable it will suffice to prove that fg is measurable on each of the sets $F_i \cap G_j$.

Observe that fg is constant on the set $F_1 \cap G_1 = f^{-1}(-\infty) \cap g^{-1}(-\infty)$. Indeed fg is constant on all of the sets $F_1 \cap G_j$, $j = 1, \ldots, 5$. Similarly, fg is constant on $F_5 \cap G_j$, $j = 1, \ldots, 5$ and $F_i \cap G_1$, $i = 1, \ldots 5$ and $F_i \cap G_5$, $i = 1, \ldots, 5$.

This leaves the nine sets $F_i \cap G_j$, $i, j = 2, 3, 4$ to consider. But the union of these sets is $S = f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$. Of course, f and g are real valued on S and so fg is measurable on S by Proposition 2.6 (p. 45). This completes the solution.

Second Solution We try to follow the proof of Proposition 2.6 on page 45 of the book. Thus, we define $F \colon X \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by $F(x) = (f(x), g(x))$. As discussed in the book, this is measurable.

Next we define $\psi \colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by $\psi(x, y) = xy$ (with the $0 \cdot (\pm\infty) = 0$ convention). Note that ψ is **not** continuous at the four points $(0, \pm\infty)$, $(\pm\infty, 0)$.

Nonetheless, we claim that ψ is Borel measurable. To see this, define $A \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by

$$A = \{ (0, -\infty), (0, \infty), (-\infty, 0), (\infty, 0) \}.$$

This is a closed subset of $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$, hence a Borel subset. Thus, $B = \overline{\mathbb{R}} \times \overline{\mathbb{R}} \setminus A$ is Borel.

Recall from our discussion in class that if Z is a metric space, and W is a Borel subset of Z,

$$(\mathcal{B}_Z)|_W = \{ E \subseteq W \mid E \in \mathcal{B}_Z \} = \mathcal{B}_W.$$

Thus, the principal of the Proposition we proved above applies: If Z is the union of two Borel sets V and W then $f \colon Z \to \overline{\mathbb{R}}$ is Borel measurable if and only if f is Borel measurable on V and W.

Thus, in order to show that ψ is Borel measurable, it will suffice to show it is Borel measurable on A and B. On A, ψ is constant (with value 0) and so Borel measurable. On B, ψ is continuous, and hence Borel measurable. This proves the claim that ψ is Borel measurable.

To complete the solution, let $h = fg$. Then $h = \psi \circ F$. Let $B \subseteq \overline{\mathbb{R}}$ be a Borel set. Since ψ is Borel measurable, $\psi^{-1}(B)$ is Borel in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Since F is measurable, $F^{-1}(\psi^{-1}(B)) \in \mathcal{M}$. Thus,

$$h^{-1}(B) = (\psi \circ F)^{-1}(B) = F^{-1}(\psi^{-1}(B)) \in \mathcal{M},$$

which shows that h is measurable.

b. Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x)+g(x)$ otherwise. Then h is measurable.

Answer:

First Solution. We can partition X into the measurable sets

$$f^{-1}(-\infty) \cap g^{-1}(-\infty) \tag{A.1}$$

$$f^{-1}(-\infty) \cap g^{-1}(\mathbb{R}) \tag{A.2}$$

$$f^{-1}(-\infty) \cap g^{-1}(\infty) \tag{A.3}$$

$$f^{-1}(\mathbb{R}) \cap g^{-1}(-\infty) \tag{A.4}$$

$$f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) \tag{A.5}$$

$$f^{-1}(\mathbb{R}) \cap g^{-1}(\infty) \tag{A.6}$$

$$f^{-1}(\infty) \cap g^{-1}(-\infty) \tag{A.7}$$

$$f^{-1}(\infty) \cap g^{-1}(\mathbb{R}) \tag{A.8}$$

$$f^{-1}(\infty) \cap g^{-1}(\infty) \tag{A.9}$$

and it will suffice to prove that h is measurable on each of these sets. On the sets (A.3) and (A.7), h is constant, with value a, by our definition. On the sets (A.1), (A.2), (A.4), (A.6), (A.8), and (A.9), h is constant (with value either $\pm\infty$). Finally, on the set $f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ in (A.5), both f and g are real valued and h coincides with $f + g$, which is measurable by Proposition 2.6. This completes the solution.

Second Solution. As in the first part of the problem, define $F\colon X \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by $F(x) = (f(x), g(x))$. As discussed in the book, this is measurable. Let $A \subseteq \overline{\mathbb{R}}$ be defined by

$$A = \{\, (-\infty, \infty), (\infty, -\infty) \,\},$$

which is closed, and hence Borel, in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Let $B = \overline{\mathbb{R}} \times \overline{\mathbb{R}} \setminus A$, which is also Borel. Define $\psi \colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by

$$\psi(x, y) = \begin{cases} a, & (x, y) \in A \\ x + y, & (x, y) \in B. \end{cases}$$

This function is **not** continuous at the points in A, but it is nonetheless Borel measurable. Of course, ψ is continuous, hence Borel measurable, on B, and ψ is constant, hence Borel measurable, on A. Thus, ψ is Borel measurable on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

Since ψ is Borel measurable, $\psi \circ F$ is measurable (as discussed in the solution of the first part of the problem), but $h = \psi \circ F$, so the proof is complete.

Problem 8. **[Problem 3, page 48]**

Let $(X, \mathcal{M})$ be a measurable space.

If $\{ f_n \}$ is a sequence of measurable functions on X, then $\{ x \mid \lim f_n \text{ exists} \}$ is a measurable set.

Answer:

The problem is not stated very well, since it leaves some ambiguity about where the values of the f_n's are supposed to be. Going by the previous problems, I'd say we should consider functions with values in $\overline{\mathbb{R}}$.

Define $E = \{ x \mid \lim f_n \text{ exists} \}$. By Proposition 2.7, the functions $g, h \colon X \to \overline{\mathbb{R}}$ defined by

$$g(x) = \liminf_{n \to \infty} f_n(x)$$
$$h(x) = \limsup_{n \to \infty} f_n(x)$$

are measurable and, of course,

$$E = \{ x \in X \mid g(x) = h(x) \},$$

so it will suffice to prove that the set where these two measurable functions are equal is measurable.

For a first attempt, one could try to define $\varphi(x) = h(x) - g(x)$, claim that φ is measurable and observe that $E = \varphi^{-1}(0)$. It's not quite that easy, since there may be points where $h(x) - g(x)$ is undefined, e.g., if $h(x) = g(x) = \infty$.

We can take care of this in the spirit of some previous solutions. We can partition X into the following measurable sets

(A.1) $$S_1 = g^{-1}(-\infty) \cap h^{-1}(-\infty)$$

(A.2) $$S_2 = g^{-1}(-\infty) \cap h^{-1}(\mathbb{R})$$

(A.3) $$S_3 = g^{-1}(-\infty) \cap h^{-1}(\infty)$$

(A.4) $$S_4 = g^{-1}(\mathbb{R}) \cap h^{-1}(-\infty)$$

(A.5) $$S_5 = g^{-1}(\mathbb{R}) \cap h^{-1}(\mathbb{R})$$

(A.6) $$S_6 = g^{-1}(\mathbb{R}) \cap h^{-1}(\infty)$$

(A.7) $$S_7 = g^{-1}(\infty) \cap H^{-1}(-\infty)$$

(A.8) $$S_8 = g^{-1}(\infty) \cap h^{-1}(\mathbb{R})$$

(A.9) $$S_9 = g^{-1}(\infty) \cap h^{-1}(\infty)$$

It will suffice to show that $E \cap S_j$ is measurable for $j = 1, \ldots, 9$ (since then E is a finite union of measurable sets).

In case (A.1), we have $E \cap S_1 = S_1$, which is measurable.

In case (A.2), $E \cap S_2 = \emptyset$, since $g(x) \neq h(x)$ for all $x \in S_2$.

In case (A.3), $E \cap S_3 = \emptyset$.

In case (A.4), $E \cap S_4 = \emptyset$.

In case (A.5), both g and h are real-valued on S_5, so $E \cap S_5 = (h - g)^{-1}(0)$, which is measurable, since $h - g$ is measurable on S_5 by Proposition 2.6.

In case (A.6), $E \cap S_6 = \emptyset$.

In case (A.7), $E \cap S_7 = \emptyset$.

In case (A.8), $E \cap S_8 = \emptyset$.

Finally, in case (A.9), $E \cap S_9 = S_9$, which is measurable.

This completes the solution.

Alternate Solution. Just for fun, here's another way to do it. Since $g \leq h$, X is the disjoint union of $E = \{\, x \mid g(x) = h(x) \,\}$ and $F = \{\, x \mid g(x) < h(x) \,\}$. Since the complement of a measurable set is measurable, it will suffice to show that F is measurable.

We claim

(∗) $$F = \bigcup_{r \in \mathbb{Q}} \{\, x \mid g(x) < r \,\} \cap \{\, x \mid r < h(x) \,\}.$$

To see this, suppose first that $p \in F$. Then $g(p) < h(p)$ so there is some rational s so that $g(p) < s < h(p)$. But then

$$p \in \{\, x \mid g(x) < s \,\} \cap \{\, x \mid s < h(x) \,\},$$

which is one of the sets in the union in (∗).

Conversely, if p is in the union in (∗), then there is some $r \in \mathbb{Q}$ so that

$$p \in \{\, x \mid g(x) < r \,\} \cap \{\, x \mid r < h(x) \,\}.$$

But then $g(p) < r$ and $r < h(p)$, so $g(p) < h(p)$, i.e., $p \in F$. This completes the proof of the claim.

Of course, each of the sets

$$\{ x \mid g(x) < r \} \cap \{ x \mid r < h(x) \} = g^{-1}([-\infty, r)) \cap h^{-1}((r, \infty])$$

is measurable, so $(*)$ shows that F is a countable union of mensurable sets, hence measurable.

Problem 9. [Problem 4, page 48]

Let $(X, \mathcal{M})$ be a measurable space.

If $f\colon X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Answer:
As remarked in the book on page 45, the Borel algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ of $\overline{\mathbb{R}}$ is generated by the rays $(a, \infty]$ for $a \in \mathbb{R}$. Hence, to show f is measurable, it will suffice to show that $f^{-1}((a, \infty])$ is measurable for each $a \in \mathbb{R}$.

To do this, let $a \in \mathbb{R}$ be fixed but arbitrary. Since the rationals are dense in $\mathbb{R}$, we can find a sequence of rationals $\{ r_n \}$ that decrease to a, i.e., $r_n \to a$ and the r_n's form a decreasing sequence.

We claim that

$$(*) \qquad (a, \infty] = \bigcup_{n=1}^{\infty} (r_n, \infty].$$

To see this, first suppose $x \in (a, \infty]$. Then $x > a$ and we can find an open interval U around a that does not contain x. Since $r_n \searrow a$, there is some N such that $r_n \in U$ for $n \geq N$. But then $r_n < x$, so $x \in (r_n, \infty]$, for $n \geq N$. Thus, x is in the union on the right hand side of $(*)$.

Conversely, if x is in the union in $(*)$, then $x \in (r_n, \infty]$ for some n. But then $a < r_n < x$, so $x \in (a, \infty]$. This completes the proof of the claim.

From $(*)$, we conclude that

$$f^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}((r_n, \infty]).$$

Since r_n is rational, $f^{-1}((r_n, \infty])$ is measurable by our hypothesis. Thus, $f^{-1}((a, \infty])$ is a countable union of measurable sets, and so is measurable. This completes the proof.

Problem 10. [Problem 9, page 48]

Let $f\colon [0, 1] \to [0, 1]$ be the Cantor function and let $g(x) = f(x) + x$.

a. g is a bijection from $[0, 1]$ to $[0, 2]$ and $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.

Answer:

Since the Cantor function is continuous, g is continuous.

Recall that the Cantor function is nondecreasing, i.e., if $x < y$ then $f(x) \leq f(y)$. But then if $x < y$, $g(x) = f(x) + x < f(y) + y = g(y)$. Thus, g is strictly increasing and hence one-to-one. Since $f(0) = 0$ and $f(1) = 1$, we have $g(0) = 0$ and $g(1) = 2$. If $x \in [0, 1]$, $0 = g(0) < g(x) < g(1) = 2$, so g maps $[0, 1]$ into $[0, 2]$. By the intermediate value theorem, every point in $[0, 2]$ is in the image of g. Thus, g is a bijection from $[0, 1]$ to $[0, 2]$.

The fact that $h = g^{-1}$ is continuous is a general fact about strictly monotone functions that follows from the intermediate value theorem. For completeness, we'll give a proof here, but you should probably skip it on a first reading.

Lemma 0.2. *Let $f: I \to \mathbb{R}$ be a continuous strictly increasing function, where $I \subseteq \mathbb{R}$ is an interval. Then the image of an interval $J \subseteq I$ is an interval of the same type (i.e., open, closed, etc.) In particular, $f(I)$ is an interval of the same type as I.*

Proof of Lemma. Let $(a, b) \subseteq I$ be an open interval. If $x \in (a, b)$ then $a < x < b$ and $f(a) < f(x) < f(b)$, since f is strictly increasing. Thus, $f((a, b)) \subseteq (f(a), f(b))$. On the other hand, if $y \in (f(a), f(b))$ then there is an $x \in (a, b)$ such that $f(x) = y$, by the intermediate value theorem. Thus, f maps the open interval (a, b) onto the open interval $(f(a), f(b))$.

Suppose that $[a, b) \subseteq I$ is a half-open interval. By the previous part of the proof (a, b) gets mapped onto $(f(a), f(b))$, and a certainly gets mapped onto $f(a)$. Thus, f maps $[a, b)$ onto $[f(a), f(b))$. The other types of intervals are dealt with in a similar fashion.

□

Proposition 0.3. *Let $f: I \to \mathbb{R}$ be a continuous strictly increasing function, where $I \subseteq \mathbb{R}$ is an interval, and let $J = f(I)$. Then the inverse function $f^{-1}: J \to I$ is continuous.*

Proof. We need to show that if $U \subseteq I$ is open relative to I, then $(f^{-1})^{-1}(U) = f(U)$ is open relative to J. Every open set in J can be written as a union of bounded relatively open intervals, and the bijection f preserves unions, so it will suffice to show that the image of a bounded relatively open interval is open in J.

If $(a, b) \subseteq I$, then (a, b) is open relative to I, and $f((a, b)) = (f(a), f(b))$ (by the lemma) which is open relative to J. If I has a lower endpoint, call it a, then intervals of the form $[a, b) \subseteq I$ are open relative to I. By the lemma,

$f(a)$ is the lower endpoint of J, and $f([a,b)) = [f(a), f(b))$, which is open relative to J. The case where I has an upper endpoint is dealt with similarly.

□

There are similar results (with the obvious modifications) for strictly decreasing functions.

b. If C is the Cantor set, $m(g(C)) = 1$.

Answer:

Recall that $A = [0,1] \setminus C$ is the union of a countably infinite collection of disjoint open intervals $\{ I_k \}_{k=1}^{\infty}$ such that

$$m([0,1] \setminus C) = \sum_{k=1}^{\infty} m(I_k) = 1$$

(which is why C has measure zero). Also recall that the Cantor function is defined to be constant on each of the closed intervals $\overline{I_k}$.

If $I_k = (a_k, b_k)$, then Cantor function f takes some constant value α on $[a_k, b_k]$. Then we have

$$\begin{aligned} g((a_k, b_k)) &= (g(a_k), g(b_k)) \\ &= (a_k + f(a_k), b_k + f(b_k)) \\ &= (a_k + \alpha, b_k + \alpha), \end{aligned}$$

which is an interval of the same length as I_k. Thus,

$$m(g(A)) = \sum_{k=1}^{\infty} m(g(I_k)) = \sum_{k=1}^{\infty} m(I_k) = 1.$$

Since $[0,1] = A \cup C$ (disjoint union), we have

$$[0,2] = g([0,1]) = g(A) \cup g(C), \qquad \text{(disjoint union)}$$

and so

$$2 = m([0,2]) = m(g(A)) + m(g(C)) = 1 + m(g(C)),$$

from which we conclude that $m(g(C)) = 1$.

c. By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

Answer:

Since $A \subseteq g(C)$, we have $B = g^{-1}(A) \subseteq C$. Thus, B is a subset of the Lebesgue nullset C, and so is Lebesgue measurable (since Lebesgue measure is complete). On the other hand, if B was Borel, so would be $g(B) = (g^{-1})^{-1}(B)$, since g^{-1} is continuous. But $g(B) = A$ and A is not Lebesgue measurable, let alone Borel. Thus, B is not Borel.

d. There exists a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ so that $F \circ G$ is not Lebesgue measurable.

Answer:

Let the sets $A \subseteq [0,2]$ and $B \subseteq C \subseteq [0,1]$ be as in the last part of the problem.

Let $F = \chi_B$, which is Lebesgue measurable since B is Lebesgue measurable and let $G\colon [0,2] \to [0,1]$ be g^{-1}, which is continuous. The function $F \circ G\colon [0,2] \to \mathbb{R}$ takes only the values 0 and 1 and

$$\begin{aligned} 1 = (F \circ G)(x) &\iff 1 = (\chi_B \circ g^{-1})(x) \\ &\iff 1 = \chi_B(g^{-1}(x)) \\ &\iff g^{-1}(x) \in B \\ &\iff x \in g(B) = A \\ &\iff 1 = \chi_A(x) \end{aligned}$$

Thus, $F \circ G = \chi_A$, which is **not** Lebesgue measurable, since the set A is nonmeasurable.

Actually, this example doesn't quite fulfill the requirements of the problem, since G is defined only on $[0,2]$, not all of $\mathbb{R}$. To fix this, extend the Cantor function f from $[0,1]$ to $\mathbb{R}$ by defining $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 1$. Then define g on $\mathbb{R}$ by $g(x) = f(x) + x$. Now g is a strictly increasing function $\mathbb{R} \to \mathbb{R}$, so $g^{-1}\colon \mathbb{R} \to \mathbb{R}$ is continuous. On $[0,1]$ this g agrees with our old g, so the above example will work with $G = g^{-1}\colon \mathbb{R} \to \mathbb{R}$ and $F = \chi_B$ (defined on all of $\mathbb{R}$).
