
EXAM

Exam #3

Math 1430, Spring 2002

April 21, 2001

ANSWERS

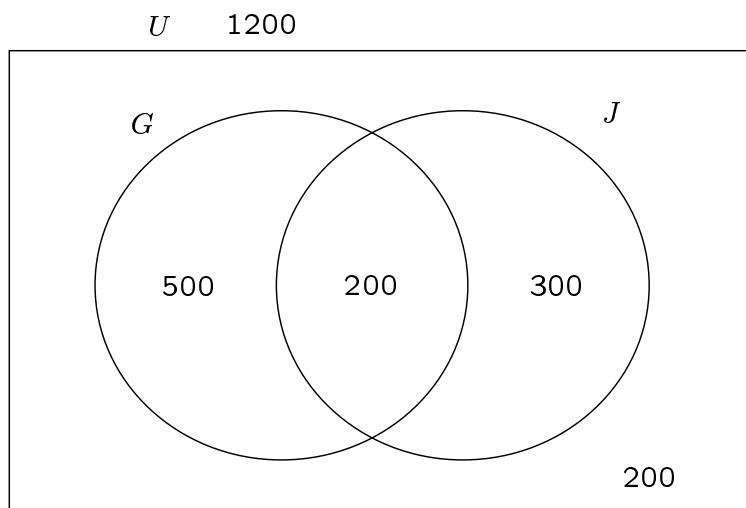
60 pts.

Problem 1. A city has two newspapers, the *Gazette* and the *Journal*. In a survey of 1,200 residents, 500 read the *Journal*, 700 read the *Gazette* and 200 read both papers.

A. How many of the residents in the survey read the *Gazette* or the *Journal*?

Answer:

Here is the Venn Diagram.



Here U is the universal set, i.e., the set of all residents surveyed. We use G for the set of residents who read the *Gazette* and J for the set of residents who read the *Journal*.

We are given that 200 residents read both papers, so we put that figure in the intersection of the two circles. The 700 people who read the *Gazette* includes the 200 that read both, so we put $500 = 700 - 200$ in the other part of the G circle. Similarly, the 500 people who read the *Journal* include the 200 who read both, so we put 300 in the other part of the J circle.

The the people who read the *Gazette* or the *Journal* are all those in the G circle or the J circle (or both), so the number is $500 + 200 + 300 = \boxed{1000}$.

B. How many of the residents in the survey read neither of the newspapers?

Answer:

Since 1000 people read one or both papers, the number who don't read a paper is $1200 - 1000 = \boxed{200}$.

- C. How many of the residents in the survey read the *Gazette* but not the *Journal*?

Answer:

These are the people in the G circle but not in the J circle, which is $\boxed{500}$ from our diagram.

60 pts.

Problem 2. A state wants to use automobile license numbers consisting of a group of three letters (A–Z) followed by a group of 3 digits (0–9).

- A. How many license plate numbers are possible if letters and digits may be repeated?

Answer:

There are three slots that can be filled in with any of 26 letters and 3 slots that can be filled in with any of 10 digits. Thus, the total number of ways of filling in the slots is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^3 = \boxed{17,576,000}.$$

- B. How many license plate numbers are possible if repeated letters and repeated digits are not allowed?

Answer:

There are 26 choices for the first letter, but only 25 for the second, since we can't choose the letter in the first slot. There are 24 choices for the third letter slot, since we can't use the two letters in the first two slots.

Similarly, there are 10 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit. Thus, by the multiplication principle, the number of ways of filling in the slots is

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = \boxed{11,232,000}.$$

80 pts.

Problem 3. A club has 10 members, 6 of whom are women and 4 of whom are men.

- A. In how many ways can the club choose a President, Vice President and Secretary?

Answer:

Since the order matters, this is the number of permutations of 3 objects out of 10,

$$P_{10,3} = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = \boxed{720}.$$

- B. If the President, Vice President and Secretary are chosen at random, with all members equally likely to be chosen, what is the probability that all of the 3 officers will be women?

Answer:

The number of ways of choosing the 3 officers from the 6 women is $P_{6,3} = 120$ (order matters). Thus, the probability that all the officers will be women is the number of ways of choosing all female officers divided by the number of possible ways of choosing the officers from the previous part of the problem. Thus, the probability that all the officers will be women is

$$\frac{P_{6,3}}{P_{10,3}} = \frac{120}{720} = \boxed{\frac{1}{6} \approx 0.1667}.$$

- C. How many ways can the club choose a committee of 4 members?

Answer:

Since all members of the committee are equal, order does not matter so the number of ways of choosing the committee is $C_{10,4}$, the number of combinations of 10 objects taken 4 at a time.

$$C_{10,4} = \frac{10!}{6!4!} = \boxed{210}.$$

- D. If all members are equally likely to be chosen, what is the probability that the 4 member committee will be made up entirely of men?

Answer:

There is only one way of forming a Committee of all men (i.e., $C_{4,4} = 1$), so the probability that the committee will be all male is

$$\frac{1}{C_{10,4}} = \boxed{\frac{1}{210} \approx 0.004762}.$$

80 pts.

Problem 4. A single card is drawn at random from a standard deck of 52 cards.

A. What is the probability that a red card is drawn?

Answer:

Let R be the event that a red card is drawn. There are two red suits of 13 cards each, so there are 26 ways of getting a red card. There are 52 cards altogether, so the probability of getting a red card is

$$P(R) = \frac{26}{52} = \frac{1}{2} .$$

B. What is the probability that a face card is drawn?

Answer:

Let F be the event that a face card is drawn. There are 3 face cards (Jack, Queen, King) in each suit and there are 4 suits, so there are 12 face cards in the deck. Thus, the probability of drawing a face card is

$$P(F) = \frac{12}{52} \approx 0.2308 .$$

C. What is the probability that a red face card is drawn?

Answer:

This is the event of getting a red card and a face card, i.e., $R \cap F$. There are 3 face cards in a suit and 2 red suits, so there are 6 red face cards. Thus,

$$P(R \cap F) = \frac{6}{52} \approx 0.1154 .$$

D. What is the probability of drawing a red card or a face card?

Answer:

Getting a red card or a face card is the event $R \cup F$. By the Union Rule,

$$\begin{aligned} P(R \cup F) &= P(R) + P(F) - P(R \cap F) \\ &= \frac{26}{52} + \frac{12}{52} - \frac{6}{52} \\ &= \frac{32}{52} \approx 0.6154 . \end{aligned}$$

100 pts.

Problem 5. A five card hand is drawn from a standard deck of 52 cards.

A. How many 5 card hands are possible?

Answer:

Since the order of the cards in the hand does not matter the number of hands is

$$\boxed{C_{52,5} = 2,598,960} .$$

B. What is the probability that all the cards in the hand drawn are hearts?

Answer:

There are 13 hearts in the deck, so the number of 5 cards hands with all hearts is $C_{13,5}$. Thus the probability of getting all hearts is

$$\boxed{\frac{C_{13,5}}{C_{52,5}} \approx 0.0004952} .$$

C. What is the probability that the hand drawn contains no hearts?

Answer:

There number of cards that are not hearts is $52 - 13 = 39$. Thus, the number of hands that contain no hearts is $C_{39,5}$. Thus, the probability of getting a hand with no hearts is

$$\boxed{\frac{C_{39,5}}{C_{52,5}} \approx 0.2215} .$$

D. What is the probability that the hand drawn contains exactly 2 hearts?

Answer:

We have to figure out how many hands contain exactly two hearts. We can break the process of choosing such a hand into two steps: First choose the two hearts, then choose 3 non-hearts.

The number of ways of choosing the 2 hearts is $C_{13,2}$. The number of ways of choosing the 3 non-hearts is $C_{39,3}$. The number of ways of performing step 1 followed by step 2 is then $C_{13,2}C_{39,3}$ by the multiplication principle.

Thus, there are $C_{13,2}C_{39,3}$ hands with exactly 2 hearts. Thus, the probability of drawing a hand with exactly two hearts is

$$\boxed{\frac{C_{13,2}C_{39,3}}{C_{52,5}} \approx 0.2743} .$$

E. What is the probability that the hand drawn contains 1 or more hearts?

Answer:

One way to do it is to note that the probability of 1 or more hearts would be the probability of exactly 1 heart plus the probability of exactly 2 hearts plus the probability of exactly 3 hearts plus the probability of exactly 4 hearts plus the probability of exactly 5 hearts.

That's too much work. An easier way is to note that this event can be described as the event of getting *some* hearts, so the complementary event is getting no hearts, which we've already figured out. Thus, the probability of getting 1 or more hearts is

$$1 - \frac{C_{39,5}}{C_{52,5}} \approx 0.7785 .$$

80 pts.

Problem 6. Mathematics student Bob is a poor driver. On the day Bob drives to his math final, the weather may be Good, Rain or Snow. Based on past performance we have the following probabilities that Bob will have an accident on the way to the final.

	Good Weather <i>G</i>	Rain <i>R</i>	Snow <i>S</i>	Totals
Accident <i>A</i>	0.1	0.1	0.2	0.4
No Accident <i>A'</i>	0.4	0.1	0.1	0.6
Totals	0.5	0.2	0.3	1.0

A. What is the probability that Bob will have an accident (no information on the weather is given)?

Answer:

This is the total for the Accident row. Thus,

$$P(A) = 0.4 .$$

B. What is the probability of snow?

Answer:

This is the total in the Snow column, so

$$P(S) = 0.3 .$$

C. What is the probability of an accident given snow?

Answer:

The intersection of the Accident row and the Snow column gives the probability of an Accident and Snow. Thus, $P(A \cap S) = 0.2$. We know that $P(S) = 0.3$. We want to know $P(A | S)$, the probability of an Accident given Snow. The formula for conditional probability gives

$$P(A | S) = \frac{P(A \cap S)}{P(S)} = \frac{0.2}{0.3} = \frac{2}{3},$$

so we get

$$\boxed{P(A | S) = 2/3 \approx 0.6667}.$$

D. Are the events accident (A) and snow (S) independent?

Answer:

The events will be independent if one does not affect the other. Thus, the events are independent if and only if $P(A | S) = P(A)$. Since we've found $P(A | S) \approx 0.6667$ and $P(A) = 0.4$, this equation does not hold, so **the events are not independent**.

An alternative approach is to recall that the events A and S are independent if and only if

$$P(A \cap S) = P(A)P(S).$$

Since we've found $P(A \cap S) = 0.2$ and we have $P(A)P(S) = (0.4)(0.3) = 0.12$, we see again that the events are not independent.

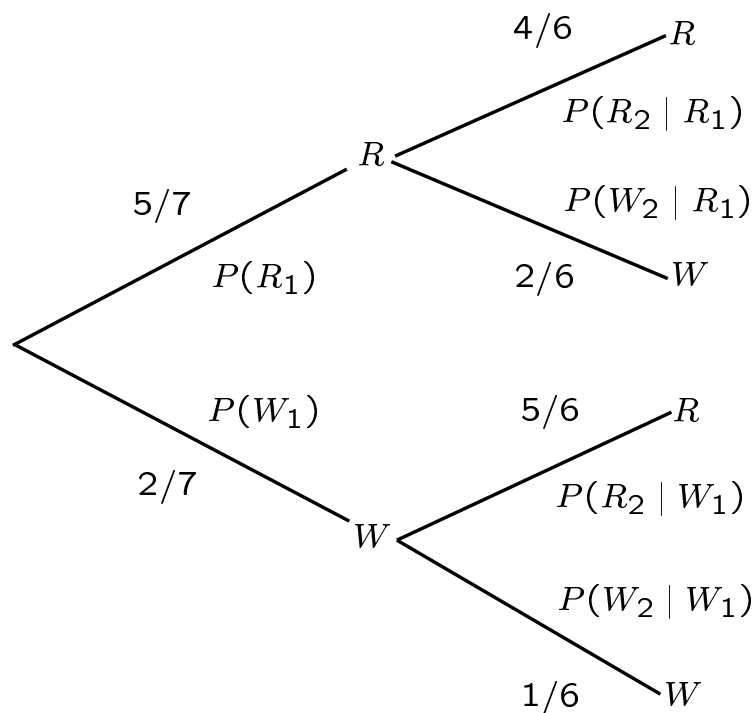
110 pts.

Problem 7. An urn contains 5 red balls and 2 white balls. Two balls are drawn successively without replacement.

A. Draw the probability tree for this experiment.

Answer:

I'll use the notation R_1 for the event of getting a red ball on the first draw, W_2 for getting a white ball on the second draw, and so forth. Here's the probability tree.



I've labeled the branches with the notation for the probabilities they represent, but this is not necessary to receive full credit.

- B. What is the probability of getting a white ball on the second draw given that the first ball drawn was red?

Answer:

$$P(W_2 | R_1) = \frac{2}{6}.$$

To get this, note that if a red ball is drawn first, on the second draw there are 4 red and 2 white balls, so the probability of getting a white ball on the second draw is $2/6$. The other probabilities in the tree are derived by similar reasoning.

- C. What is the probability of getting a white ball on the second draw?

Answer:

This would be the sum of the products along the branches ending in W . In

formulas,

$$\begin{aligned} P(W_2) &= P(R_1 \cap W_2) + P(W_1 \cap W_2) \\ &= P(W_2 | R_1)P(R_1) + P(W_2 | W_1)P(W_1) \\ &= \frac{2}{6} \frac{5}{7} + \frac{1}{6} \frac{2}{7} \\ &\approx \boxed{0.2857}. \end{aligned}$$

D. What is the probability that both balls drawn have the same color?

Answer:

This is the sum of the products along the branches where both balls have the same color. In formulas

$$\begin{aligned} P(\text{same color}) &= P((R_1 \cap R_2) \cup (W_1 \cap W_2)) \\ &= P(R_1 \cap R_2) + P(W_1 \cap W_2) \\ &= P(R_2 | R_1)P(R_1) + P(W_2 | W_1)P(W_1) \\ &= \frac{4}{6} \frac{5}{7} + \frac{1}{6} \frac{2}{7} \\ &\approx \boxed{0.5238}. \end{aligned}$$

E. What is the probability that a white ball was drawn on the first draw, given that a red ball was drawn on the second draw. Hint: Bayes' Formula.

Answer:

We want $P(W_1 | R_2)$. According to Bayes' Formula, this is the product along the branch through W to R , divided by the sum of the products along the branches that end in R . Thus,

$$P(W_1 | R_2) = \frac{\frac{2}{6} \frac{5}{7}}{\frac{2}{6} \frac{5}{7} + \frac{5}{6} \frac{4}{7}} = \boxed{\frac{1}{3} \approx 0.3333}.$$

In formulas this is

$$P(W_1 | R_2) = \frac{P(R_2 | W_1)P(W_1)}{P(R_2 | W_1)P(W_1) + P(R_2 | R_1)P(R_1)}.$$