EXAM

Exam #2

Math 1430, Spring 2002

March 26, 2001

ANSWERS

Here are some possibly useful formulas from Chapter 3:

$$A = P(1 + rt), \qquad A = P(1 + i)^n,$$

$$\text{APY} = \left(1 + \frac{r}{m}\right)^m - 1, \qquad \text{FV} = \text{PMT}\,\frac{(1+i)^n - 1}{i},$$

$$\text{PV} = \text{PMT}\,\frac{1 - (1+i)^{-n}}{i}.$$

cd Give dollar amounts to the nearest cent.

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40 pts.

Problem 1. Find the equation of the line that passes through the points $(-2, 4)$ and $(1, -2)$. Give the equation of the line in slope-intercept form. Find the x-intercept and y-intercept of the line.

Answer:
Let $(x_1, y_1) = (-2, 4)$ and $(x_2, y_2) = (1, -2)$. Using the formula for the slope of a line, we have

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 4}{1 - (-2)} \\ &= \frac{-6}{3} \\ &= -2 \end{aligned}$$

If a line has slope m and the point (x_0, y_0) is on the line, the point-slope form of the equation of the line is

$$y - y_0 = m(x - x_0).$$

In our case, $m = -2$ and we can choose $(x_0, y_0) = (1, -2)$ (you could choose the other point just as well). Plugging into the point-slope form, we get

$$y - (-2) = -2(x - 1).$$

This simplifies to

$$y + 2 = -2x + 2$$

or just

$$y = -2x$$

This is in the slope-intercept form $y = mx + b$—in our case $b = 0$. Thus, our answer for the equation of the line in slope-intercept form is

$$\boxed{y = -2x}\,.$$

In the slope-intercept form $y = mx + b$, b is the y-intercept, so in our case the y-intercept is $y = 0$. To find the x-intercept, set y equal to zero in the equation $y = -2x$ of the line and solve for x. The solution is $x = 0$. Thus, we have

$$\boxed{\begin{array}{c} x\text{-intercept} = 0 \\ y\text{-intercept} = 0 \end{array}}$$

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40 pts. **Problem 2**. Complete the square in the equation $y = -x^2 + 2x + 3$ (i.e., find h and k so that $y = -x^2 + 2x + 3 = -(x - h)^2 + k$. Use this information to sketch the graph of $y = -x^2 + 2x + 3$. Indicate on your graph the vertex and the axis of symmetry. Find the maximum or minimum value of the function.

Answer:
For help on completing the square, see the note "How to Complete the Square" on my web page.[1] Our equation is $y = -x^2 + 2x + 3$. We first factor out the coefficient of x^2 from the first two terms to get

$$y = -(x^2 - 2x) + 3.$$

The coefficient of x inside the parentheses is $r = -2$. Thus, we have $h = -r/2 = -(-2)/2 = 1$ and $h^2 = 1^2 = 1$. We next add and subtract h^2 inside the parentheses to get

$$y = -(x^2 - 2x + 1 - 1) + 3.$$

We then factor the first three terms in the parentheses to get

$$y = -(\underbrace{x^2 - 2x + 1}_{(x-1)^2)} - 1) + 3 = -((x - 1)^2 - 1) + 3.$$

Finally, we simplify

$$y = -(x - 1)^2 + 1 + 3 = -(x - 1)^2 + 4.$$

From this, we see that the graph of $y = -(x - 1)^2 + 4$ is the graph of $y = -x^2$ shifted right one unit and up four units. The vertex is at $(1, 4)$ and the axis of symmetry is the vertical line $x = 1$. From the minus sign, we see that the graph is a parabola opening downward. Thus, the maxium value of the function is $\boxed{4}$.

Here is the graph.

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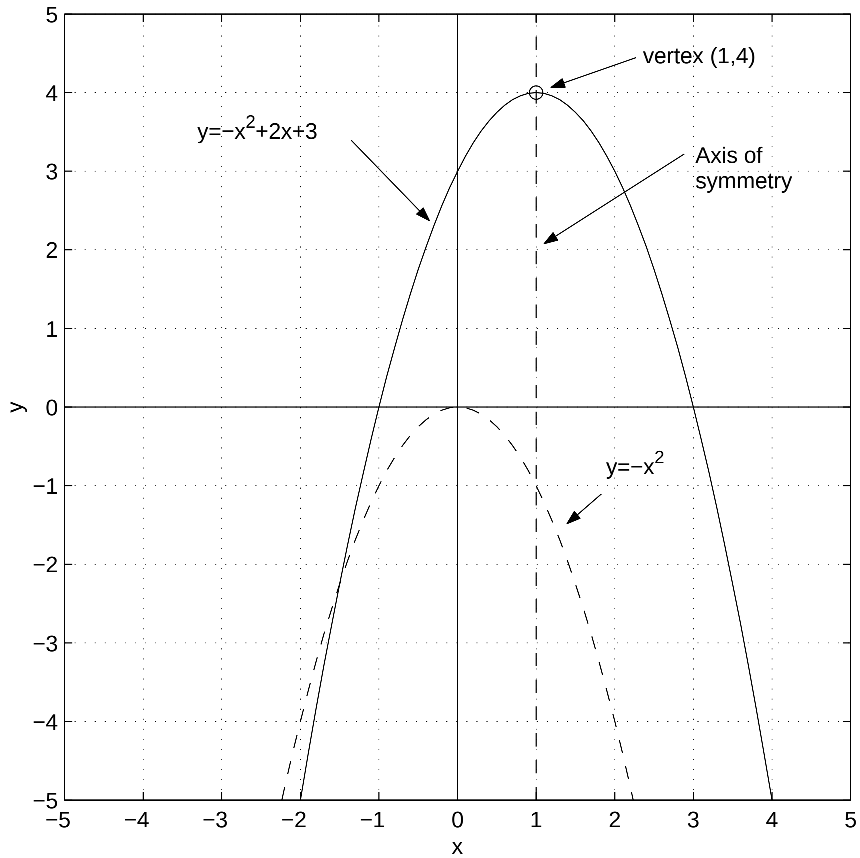
[1] `http://www.math.ttu.edu/~drager`



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40 pts.

Problem 3. The Acme corporation is a small shop that manufactures surfboards. If x denotes the number of surfboards that Acme produces per day, Acme's revenue and cost functions (in hundreds of dollars per day) are given by

$$R(x) = x(10 - x)$$
$$C(x) = 2x + 12.$$

A. Find the break even points, i.e., the production levels for which cost equals revenue.

Answer:

We want to solve the equation $R(x) = C(x)$. Plugging in the expressions above this gives

$$x(10 - x) = 2x + 12.$$

We can make the following sequence of simplifications

$$\begin{aligned} 10x - x^2 &= 2x + 12 \\ 0 &= x^2 - 10x + 2x + 12 \\ 0 &= x^2 - 8x + 12 \\ 0 &= (x-2)(x-6) \end{aligned}$$

so the solutions are $x = 2, 6$. Thus, the break even points are $\boxed{x = 2}$ and $\boxed{x = 6}$.

B. What production level will give the maximum profit? What is the maximum profit?

Answer:

We first write down the profit function $P(x) = R(x) - C(x)$. This gives

$$P(x) = x(10 - x) - (2x + 12) = -x^2 + 8x - 12.$$

To find the maximum, we complete the square. The first step[2] is to factor the coefficient of x^2 out of the first two terms. This gives

$$P(x) = -(x^2 - 8x) - 12.$$

The coefficient of x inside the parentheses is $r = -8$ Thus, $h = -r/2 = -(-8)/2 = 4$ and $h^2 = 16$. Next, we add and subtract h^2 inside the parentheses. This gives

$$P(x) = -(x^2 - 8x + 16 - 16) - 12.$$

Next, factor the first three terms in the parenthesis, so we get

$$P(x) = -(\underbrace{x^2 - 8x + 16}_{(x-4)^2} - 16) - 12 = -((x-4)^2 - 16) - 12.$$

Finally, we simplify to get

$$P(x) = -(x-4)^2 + 4.$$

From this, we see that the graph of $y = P(x)$ is a parabola that opens downward, with vertex at $(4, 4)$. Thus, maximum profit occurs at a production level of $\boxed{x = 4}$ surfboards per day and the maximum profit is $\boxed{4}$ hundred dollars per day.

[2]See "How to Complete the Square" my web page.

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40 pts. **Problem 4**. Suppose that you open a savings account with a deposit of $\$10,000$ and make no further deposits. The savings account pays 8% interest compounded quarterly. How much will be in the account at the end of 10 years?

Answer:
Since there is only one deposit and the interest is compounded, we use the compound interest formula $A = P(1+i)^n$. In this problem the principal is $P = 10,000$. The interest rate is $r = 8\% = 0.08$. The number of compounding periods per year is $m = 4$ (since we're compounding quarterly). Thus, the interest rate per period is $i = r/m = 0.08/4 = 0.02$. The number of years is $t = 10$ and the number of compounding periods per year is $m = 4$. Thus, the number of compounding periods is $n = mt = 4(10) = 40$.

If we plug all this into the formula, we get

$$\begin{aligned} A &= P(1+i)^n \\ &= 10000(1+0.02)^{40} \\ &= 10000(1.02)^{40}. \end{aligned}$$

Plugging this into a calculator gives $\boxed{A = \$22,080.40}$ (rounded off to the nearest cent).

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40 pts. **Problem 5**. Suppose that you open a savings account with a deposit of $\$10,000$ and make no further deposits. The savings account pays 6% interest, compounded monthly. How long will it take for the amount in the account to grow to $\$30,000$? (If the answer is not an even number of months, round up to the next month.)

Answer:
Since there is only one deposit and the interest is compounded, we use the compound interest formula $A = P(1+i)^n$. In this problem, the unknown is n, the number of compounding periods. For the other variables, we have $P = \$10,000$, and $A = \$30,000$. The interest rate is $r = 6\% = 0.06$. The number of compounding periods per year is $m = 12$. The interest rate per period is $i = r/m = 0.06/12 = 0.005$. Plugging all this into the formula gives

$$30000 = 10000(1.005)^n.$$

To solve this for n, first divide both sides by $10,000$ and note that $30000/10000 = 3$, thus we have

$$3 = (1.005)^n.$$

Next, take log of both sides, giving

$$\log(3) = \log\Big((1.005)^n\Big).$$

Using the power rule for logs ($\log(x^p) = p\log(x)$), the last equation becomes

$$\log(3) = n\log(1.005).$$

Finally, dividing both sides by log(1.005) gives

$$n = \frac{\log(3)}{\log(1.005)}.$$

Plugging this into a calculator gives $n \approx 220.27$. Since this is not an even number of months, we round up to the next month to get $\boxed{n = 221 \text{ months}}$ (which is 18 years and 5 months).

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40 pts.

Problem 6. Which is the better investment: A savings account that pays 8% interest compounded semiannually or a savings account that pays 7.9% interest compounded monthly? Give the annual percentage yield in both cases, to 4 decimal places.

Answer:
Compute the annual percentage yield APY in both cases, and then compare.

For the first account, we have $r = 8\% = 0.08$ and the number of compounding periods per year is $m = 2$. So, for this account,

$$\begin{aligned}\text{APY} &= (1 + r/m)^m - 1 \\ &= \left(1 + \frac{0.08}{2}\right)^2 - 1 \\ &= 0.081600 \\ &= \boxed{8.1600\%},\end{aligned}$$

using a calculator.

For the second account, $r = 7.9\% = 0.079$ and $m = 12$, so we have

$$\begin{aligned}\text{APY} &= (1 + r/m)^m - 1 \\ &= \left(1 + \frac{0.079}{12}\right)^{12} - 1 \\ &\approx 0.081924 \\ &= \boxed{8.1924\%}.\end{aligned}$$

Since the APY is higher for the second account, **the second account is the better investment**.

40 pts. **Problem 7**. A couple wishes to save \$80,000 for their children's education. They will make equal monthly payments for 20 years into an account that pays 7.5% compounded monthly. What should the monthly deposit be?

Answer:
Since we're making periodic deposits, this problem involves an annuity. Since we're building up the account for future use, we should use the future value formula

$$\mathrm{FV} = \mathrm{PMT}\,\frac{(1+i)^n - 1}{i}.$$

Since we're trying to figure out the periodic payment PMT, we solve this equation for PMT to get

$$\mathrm{PMT} = \mathrm{FV}\,\frac{i}{(1+i)^n - 1}.$$

The future value we want is $\mathrm{FV} = 80,000$. The interest rate is $r = 7.5\% = 0.075$. The number of periods per year is $m = 12$, so the interest rate per period is $i = r/m = 0.075/12 = 0.00625$. The number of years is $t = 20$ so the number of periods is $n = mt = (12)(20) = 240$. Plugging into the last equation gives

$$\begin{aligned}\mathrm{PMT} &= \mathrm{FV}\,\frac{i}{(1+i)^n - 1}\\ &= 80,000\frac{0.00625}{(1+0.00625)^{240} - 1}\\ &= \boxed{\$144.47},\end{aligned}$$

rounded to the nearest cent.

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50 pts. **Problem 8**. Suppose that you borrow \$120,000 on a 30 year mortgage to buy a house. You will make equal monthly payments and the interest rate is 7.2%, compounded monthly. What will the monthly payment be? How much interest will you pay over the life of the loan?

Answer:
Since we're amortizing a loan, we use the present value formula

$$\mathrm{PV} = \mathrm{PMT}\,\frac{1 - (1+i)^{-n}}{i}.$$

Since we want to find the payment, we solve this equation for PMT and get

$$\mathrm{PMT} = \mathrm{PV}\,\frac{i}{1 - (1+i)^{-n}}.$$

We're borrowing \$120,000, so we have PV = 120,000. The interest rate is $r = 7.2\% = 0.072$. Since we're making monthly payments, the number of periods per year is $m = 12$. The interest rate per period is $i = r/m = 0.072/12 = 0.006$. The number of years is $t = 30$, so the number of payments is $n = mt = (12)(30) = 360$. Plugging into the formula, we have

$$\begin{aligned} \text{PMT} &= \text{PV}\,\frac{i}{1-(1+i)^{-n}} \\ &= 120,000\frac{0.006}{1-(1.006)^{-360}} \\ &= \boxed{\$814.55}\,, \end{aligned}$$

rounded to the nearest cent.

The total amount of money we pay is $(360)(814.55) = 293,238.00$ and we've received \$120,000. So the total interest paid is $\$293,238.00 - \$120,00 =$ $\boxed{\$173,238.00}$.

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40 pts. **Problem 9**. Suppose that you take out a loan for \$5,000 to be repaid in 4 equal quarterly payments. The interest rate is 10% compounded quarterly. Find the amount of the quarterly payment. Construct the amortization table for this loan. (Column Headings: Payment Number, Payment Amount, Interest, Balance Reduction, Unpaid Balance.)

Answer:
We first need to figure out the monthly payment. As in the last problem,

$$\text{PMT} = \text{PV}\,\frac{i}{1-(1+i)^{-n}}.$$

The amount of the loan is PV = 5000. The interest rate is $r = 10\% = 0.1$. The number of periods per year is $m = 4$, so the interest rate per period is $i = 0.1/4 = 0.025$. Since we're making 4 payments, $n = 4$. Thus, we have

$$\begin{aligned} \text{PMT} &= \text{PV}\,\frac{i}{1-(1+i)^{-n}} \\ &= 5000\frac{0.025}{1-(1.025)^{-4}} \\ &= \boxed{\$1329.09}\,, \end{aligned}$$

rounded to the nearest cent.

To construct the amortization table, use $i = 0.025$ to compute the interest owed for each period. Here is the table:

Payment Number	Payment Amount	Interest	Balance Reduction	Unpaid Balance
0				5000.00
1	1329.09	125.00	1204.09	3795.91
2	1329.09	94.90	1234.19	2561.72
3	1329.09	64.04	1265.05	1296.67
4	1329.09	32.42	1296.67	0.00

In order to see how we get the table, here is the table with the calculations indicated

Payment Number	Payment Amount	Interest	Balance Reduction	Unpaid Balance
0				5000.00
1	13209.09	$i(5000) =$ 125.00	13209.09 − 125 = 1204.09	5000.00 − 1204.09 = 3795.91
2	13209.09	$i(3795.91) =$ 94.90	1329.09 − 94.90 = 1234.19	3795.91 − 1234.19 = 2561.72
3	13209.09	$i(2561.72) =$ 64.04	1329.09 − 64.04 = 1265.05	2561.72 − 1265.05 = 1296.67
4	32.42 + 1296.67 = 1329.09	$i(1296.67) =$ 32.42	1296.67	0.00

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40 pts.

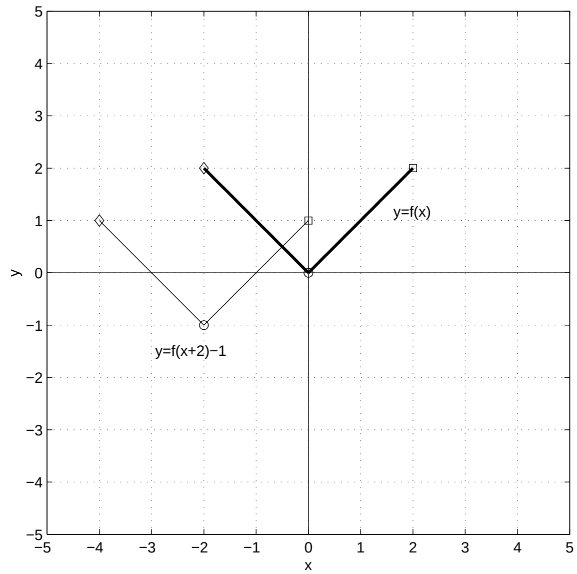
Problem 10.

A. Let $f(x)$ be the function whose graph $y = f(x)$ is drawn in the figure below. (The domain is $[-2, 2]$). On the same figure, draw the graph of $y = f(x + 2) - 1$.

Answer:

The graph of $y = f(x + 2) - 1$ is obtained by shifting the graph $y = f(x)$ two units left and one unit down.

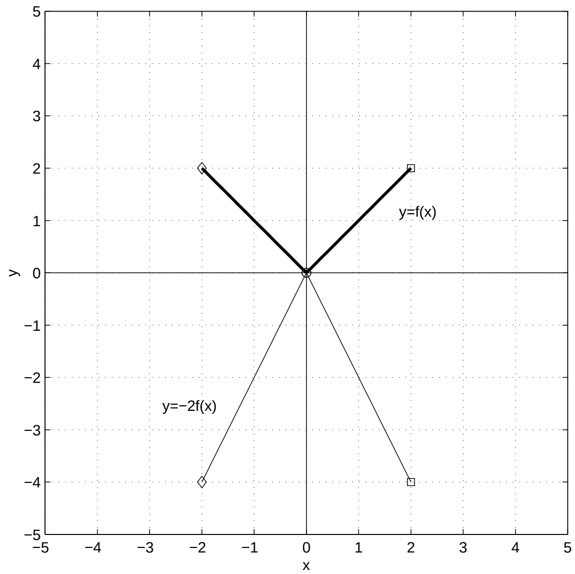
To draw this, it's handy to select a few points on the original graph, see where they go, and connect the dots. The point $(0, 0)$ on the original graph (marked with a circle on the picture below) moves left 2 and down 1 to the point $(-2, -1)$ (also marked with a circle). The point $(2, 2)$ on the original graph (square) moves to the point $(0, 1)$ (square). The point $(-2, 2)$ (diamond) moves to $(-4, 1)$ (diamond). Now connect the dots to get the new graph. Here is the picture.



B. Here is the graph $y = f(x)$ again. On this figure, draw the graph of $y = -2f(x)$.

Answer:

In this case, to get the graph of $y = -2f(x)$, we multiply the y-coordinate of every point on the original graph by -2 to get the points on the new graph. The point $(0, 0)$ (marked with a circle) moves to $(0, -2 \cdot 0) = (0, 0)$, i.e., it doesn't move. The point $(2, 2)$ on the original graph (square) moves to $(2, -2 \cdot 2) = (2, -4)$ (also marked with a square). The point $(-2, 2)$ on the original graph (diamond) moves to $(-2, -2 \cdot 2) = (-2, -4)$ (diamond). Connecting the dots gives the new graph. Here is the picture.


y=f(x)
y=−2f(x)
x
y