1. Let \( p_1, p_2, \ldots \) denote the increasing sequence of primes. Show that for any \( n \geq 1 \), the sum
\[
\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}
\]
is not an integer.

**Solution:** Certainly \( 1/2 \) is not an integer, so suppose \( n > 1 \). Let \( c_k = \frac{p_1^{2k-1}}{p_k} = p_1 p_2 \cdots p_{k-1} p_{k+1} \cdots p_n \), and notice that
\[
x = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = \frac{c_1 + c_2 + \cdots + c_n}{p_1 p_2 \cdots p_n}.
\]
Suppose by way of contradiction that \( x \in \mathbb{Z} \). It follows, in particular, that the numerator is divisible by \( p_1 \) so that
\[
c_1 + c_2 + \cdots + c_n \equiv 0 \pmod{p_1}.
\]
However, it follows from the definition of the \( c_k \)'s that \( c_2, c_3, \ldots, c_n \) are all divisible by \( p_1 \), whence
\[
0 \equiv c_1 + c_2 + \cdots + c_n \equiv c_1 \equiv p_2 p_3 \cdots p_n \pmod{p_1},
\]
a contradiction since the product on the right is clearly not divisible by \( p_1 \).

2. Show that for \( n \geq 1 \),
\[
\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.
\]

**Solution:** By the Binomial Theorem we have that
\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.
\]
The result follows immediately upon evaluating at \( a = -1, b = 1 \).

3. Show that for any positive integer \( n \),
\[
\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0,
\]
where (of course) \( \mu(n) \) is the Möbius function.

**Solution:** For any integer \( n \), the set \( \{n, n+1, n+2, n+3\} \) forms a complete system of residues modulo 4. In particular, at least one of these is congruent to zero modulo 4, whence divisible by 4. But by definition of \( \mu \), \( \mu(4k) = 0 \) for any nonnegative integer \( k \), whence the product \( \mu(n)\mu(n+1)\mu(n+2)\mu(n+3) \) is zero.

4. Compute each of the following:
   - \( d(10450) \) (where \( d(n) = \sum_{d|n} 1 \))
   - \( \sigma(10450) \)
   - \( \phi(10450) \)
   - \( \mu(10450) \)

---

1This document is copyright ©2004 Chris Monico, and may not be reproduced in any form without written permission from the author.
Solution: Notice first that $10450 = 2 \cdot 5^2 \cdot 11 \cdot 19$. Thus,

\[
d(10450) = (1 + 1)(1 + 2)(1 + 1)(1 + 1) = 24.
\]
\[
\sigma(10450) = (1 + 2)(1 + 5 + 5^2)(1 + 11)(1 + 19) = 22320.
\]
\[
\phi(10450) = (2 - 1)2^6(5 - 1)5^1(11 - 1)(19 - 1) = 3600.
\]
\[
\mu(10450) = 0, \text{ since } 5^2 \mid 10450.
\]

5. Consider the following Diophantine equation:

\[
x^2 + y^2 = 12z + 7.
\]

Does it have any integer solutions?

Solution: No, it does not. Suppose that $x, y, z$ are integers satisfying the given equation. It follows that

\[
x^2 + y^2 \equiv 7 \pmod{12} \tag{0.1}
\]

Then for any integer $a$, $a^2$ is congruent to exactly one of \{0, 1, 4, 9\} modulo 12. Thus, the sum of two squares must be congruent to one of 0, 1, 2, 4, 5, 6, 8, 9, 10 modulo 12, contradicting Equation 0.1.

6. Show that 17 is a quadratic nonresidue modulo 1009 (Hint: use quadratic reciprocity).

Solution: Both 17 and 1009 are prime, and so it follows from quadratic reciprocity that

\[
\left(\frac{17}{1009}\right) \left(\frac{1009}{17}\right) = (-1)^{(17-1)(1009-1)/4} = 1,
\]

so that $(\frac{17}{1009})$ and $(\frac{1009}{17})$ are both 1 or both -1. The latter Legendre symbol is easily computed, since

\[
\left(\frac{1009}{17}\right) = \left(\frac{1009 \mod 17}{17}\right) = \left(\frac{6}{17}\right).
\]

By brute force, we see that the quadratic residues modulo 17 are precisely \{1, 2, 4, 8, 9, 13, 15, 16\}, whence 6 is a quadratic nonresidue modulo 17 and so $(\frac{17}{1009}) = -1$.

7. Let $d(n) = \sum_{d|n} 1$. Show that $d(n)$ is odd if and only if $n$ is a perfect square.

Solution: The result clearly holds for $n = 1$, so assume $n > 1$. If $n = p_1^{e_1} \cdots p_k^{e_k}$ is the canonical factorization of $n$ into primes, we have that

\[
d(n) = (1 + e_1)(1 + e_2) \cdots (1 + e_k).
\]

If any of the $e_j$'s are odd (so that $n$ is not a perfect square), the corresponding factor in the above product is even, whence $d(n)$ is even. Conversely, if all of the $e_j$'s are even (so that $n$ is a perfect square), all of the factors in the above product are odd, whence $d(n)$ is odd.

8. Show that $4(29!) + 5!$ is divisible by 31.

Solution: Since 31 is prime, it follows from Wilson’s Theorem that

\[
-1 \equiv 30! \equiv (29!)(30) \equiv (29!)(-1) \pmod{31}.
\]

Upon multiplying both sides by -1, we see that 29! $\equiv 1 \pmod{31}$ and so

\[
4(29!) + 5! \equiv 4(1) + 120 \equiv 124 \equiv 4 \cdot 31 \equiv 0 \pmod{31}.
\]