5.1.3. Let $E = \{1/n \mid n \in \mathbb{N}\}$. Show that the function

$$f(x) = \begin{cases} 
1, & \text{if } x \in E \\
0, & \text{otherwise},
\end{cases}$$

is (Riemann) integrable on $[0, 1]$. 

Proof. $f$ is clearly bounded on $[0, 1]$. Let $\epsilon > 0$. We will construct a partition $P$ of $[0, 1]$ for which $U(f, P) - L(f, P) < \epsilon$. Let $k$ be a positive integer so that $\frac{1}{10^k} < \epsilon$. Let $P = \{\frac{1}{10^k}\} \cup \{\frac{1}{n} \pm \frac{1}{10^k} \mid n = 1, 2, \ldots, 10^k - 1\}$. Now observe that if $x \in (1/10^k, 1]$ and $f(x) \neq 0$, then $f(x) = 1$ and $x$ is contained in a subinterval of $P$ of width $2/10^{2k}$. Thus, we easily compute the upper Riemann sum by simply computing the nonzero terms,

$$U(f, P) = \sum M_j(f)(x_j - x_{j-1}) = \frac{1}{10^k} + (10^k - 1) \left(\frac{2}{10^{2k}}\right) = \frac{3 \cdot 10^k - 2}{10^{2k}} < \epsilon.$$

On the other hand, any subinterval of $P$ contains an irrational so that $m_j(f) = 0$ for $j = 1, 2, \ldots$ and the lower Riemann sum is zero. Thus,

$$U(f, P) - L(f, P) < \epsilon - 0 = \epsilon,$$

and so $f$ is Riemann integrable. Since $f$ is R.I., we have that

$$\int_0^1 f(x) \, dx = (L) \int_0^1 f(x) \, dx = 0.$$ 

5.1.4(a). If $f$ is continuous at $x_0 \in [a, b]$ and $f(x_0) \neq 0$, then

$$(L) \int_a^b |f(x)| \, dx > 0.$$ 

Before we proceed with this problem, compare with problem 3 to see the necessity of the continuity condition!

Proof. Assume that $x_0 \in (a, b)$ (the proof is adapted in a very straightforward way to the case where $x_0$ is an endpoint). Since $f$ is continuous at $x_0$, there exists a $\delta' > 0$ so that

$$|x - x_0| < \delta' \Rightarrow |f(x) - f(x_0)| < \frac{1}{2} |f(x_0)|.$$ 

Set $\delta = \min\{\delta', |x_0 - a|, |b - x_0|\}$, so that $(x_0 - \delta, x_0 + \delta) \subseteq [a, b]$. Observe now that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have $|f(x)| > (1/2)|f(x_0)|$. Let $P$ be any partition of $[a, b]$ and $P' = P \cup \{x_0 - \delta, x_0 + \delta\}$. Since $m_j(|f|) \geq 0$ for each $j$, it follows that

$$L(|f|, P') \geq 2\delta \frac{1}{2}|f(x_0)| = \delta |f(x_0)| > 0,$$

whence

$$(L) \int_a^b |f(x)| \, dx = \sup\{L(|f|, Q) \mid Q \text{ is a partition of } [a, b]\} \geq L(|f|, P') > \delta |f(x_0)| > 0.$$
5.1.6. Suppose \( f \) is R.I. on \([a, b]\) and \( E \) a finite subset of \([a, b]\). If \( g \) is a bounded function with \( g(x) = f(x) \) for all \( x \in [a, b] \setminus E \), then \( g \) is Riemann integrable on \([a, b]\) and
\[
\int_a^b g(x) \, dx = \int_a^b f(x) \, dx.
\]

**Proof.** Recall that if \( f_1, f_2 \) are R.I. on \([a, b]\), then so is \( f_1 - f_2 \). We will show that the function \( f - g \) is R.I. on \([a, b]\) and has integral zero (from which it follows that \( f - (f - g) = g \) is R.I.).

Let \( \epsilon > 0 \). Let
\[
M = \sup_{x \in E} \{|f(x) - g(x)|\},
\]
and set
\[
\delta = \min \left\{ \frac{\epsilon}{2M|E|}, \inf_{e_1, e_2 \in E} \{|e_1 - e_2|\} \right\}.
\]

Let \( P = \{e \pm \delta \mid e \in E\} \). Then notice that if \((f - g)(x_0) \neq 0\), \( x_0 \) is contained in a subinterval of width \( 2\delta \). Thus,
\[
|U(f - g, P)| = \left| \sum M_j(f - g)(x_j - x_{j-1}) \right| \leq \sum |M_j(f - g)|(x_j - x_{j-1}) \leq M|E|2\delta = \epsilon. \tag{0.1}
\]

On the other hand, since \( f(x) - g(x) \) is nonzero at only finitely many points, we obtain precisely the same bound on the lower Riemann sum,
\[
|L(f - g, P)| \leq \epsilon,
\]
whence \( U(f - g, P) - L(f - g, P) < 2\epsilon \), so that \( f - g \) is R.I. Finally, it follows from Equation 0.1 that \( \int_a^b (f - g) \, dx = 0 \). \( \square \)