A note on Turán type and mean inequalities for the Kummer function
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ABSTRACT
Turán-type inequalities for combinations of Kummer functions involving \( \Phi(a \pm \nu, c \pm \nu, x) \) and \( \Phi(a, c \pm \nu, x) \) have been recently investigated in ([Á. Baricz, Functional inequalities involving Bessel and modified Bessel functions of the first kind, Expo. Math. 26 (3) (2008) 279–293; M.E.H. Ismail, A. Laforgia, Monotonicity properties of determinants of special functions, Constr. Approx. 26 (2007) 1–9]. In the current paper, we resolve the corresponding Turán-type and closely related mean inequalities for the additional case involving \( \Phi(a \pm 1, c, x) \). The application to modeling credit risk is also summarized.

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1. Introduction

The Kummer confluent hypergeometric function is given by

\[
\Phi(\alpha, \beta, x) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{(\beta)_n n!},
\]

where \( (\alpha)_n \) is the Pochhammer symbol defined by \( (\alpha)_n \equiv \Gamma(\alpha + n)/ \Gamma(\alpha) = \alpha(\alpha + 1) \cdots (\alpha + n - 1) \) for \( n \in \mathbb{N} \), \( (\alpha)_0 = 1 \), \( (\alpha + 1)_{-1} = \frac{1}{\alpha} \), and the Gamma function is \( \Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t} \, dt \), for \( x > 0 \).

Inequalities involving contiguous Kummer confluent hypergeometric functions of the form \( \Phi(a \pm \nu, c \pm \nu, x) \) and \( \Phi(a, c \pm \nu, x) \) were presented in Theorem 2 of [4] and Theorem 2.7 of [11]. These inequalities are of the Turán type [15] in the case that \( \nu = 1 \). In the present note, we resolve the remaining Turán-type case involving \( \Phi(a \pm 1, c, x) \) and extend it to include \( \Phi(a \pm \nu, c, x) \), \( \nu \in \mathbb{N} \). We then establish a closely related mean inequality that provides simultaneous upper and lower bounds for \( \Phi(a, c, x) \). Turán-type inequalities, which are of independent interest, also have important applications in Information Theory (as demonstrated by McEliece, Reznick, and Shearer in their paper [12]) and in modeling credit risk, as summarized below.

In particular, Carey and Gordy [9] model a lending relationship in which the bank has an option to foreclose upon the borrower at any time. Following the seminal models of Merton [13] and Black and Cox [6], it is assumed that the value of the firm’s assets follows a geometric Brownian motion. It is shown that the bank’s optimal foreclosure threshold solves a first order condition involving a ratio of contiguous Kummer functions, which implies that a Turán-type inequality for the Kummer function arises naturally in studying the comparative statics of the model. A proof of this key Turán-type
inequality is established in this paper. (For general background on applications of the Kummer function in economic theory and econometrics, see [1].)

2. Main results

Theorem 1. Suppose \( a, b > 0 \). Then for any \( \nu \in \mathbb{N} \) with \( a, b \geq \nu - 1 \)

\[ \Phi(a, a+b, x)^2 > \Phi(a+\nu, a+b, x)\Phi(a-\nu, a+b, x), \]

for all nonzero \( x \in \mathbb{R} \). Moreover, these expressions coincide with value 1 when \( x = 0 \) and asymptotically for any \( x \) when \( b \to \infty \).

Corollary 2. Suppose \( a > 0 \) and \( c+1 > 0 \) with \( c \neq 0 \). Then for any \( \nu \in \mathbb{N} \) with \( a \geq \nu - 1 \),

\[ \Phi(a, c, x)^2 \geq \Phi(a-\nu, c, x)\Phi(a+\nu, c, x), \]

for all \( x > 0 \).

The next result begins with the well-known arithmetic mean-geometric mean inequality,

\[ A(\alpha, \beta) \equiv \frac{\alpha + \beta}{2} > \sqrt{\alpha\beta} \equiv G(\alpha, \beta) \quad \text{for } \alpha, \beta \text{ distinct and positive}, \]

which has many interesting refinements and applications (e.g., see [7,8]). Corollary 3 is a refinement of inequality (3) with \( \alpha = \Phi(a+\nu, a+b, x) \) and \( \beta = \Phi(a-\nu, a+b, x) \) (see illustrated special case in Fig. 1).

Corollary 3. Suppose \( \nu \in \mathbb{N} \) and \( a, b \geq \nu \). Then for all nonzero \( x \in \mathbb{R} \)

\[ A(\Phi(a+\nu, a+b, x), \Phi(a-\nu, a+b, x)) > \Phi(a, a+b, x) \]

\[ > G(\Phi(a+\nu, a+b, x), \Phi(a-\nu, a+b, x)). \]

It is also interesting to compare these results with the elegant Theorem 2.3 and open problems in [5] regarding Turán-type and arithmetic mean-geometric mean inequalities involving the Gaussian hypergeometric function \(_2F_1\).

3. Proofs

Proof of Theorem 1. First assume \( x > 0 \). For \( c > -1, c \neq 0 \), define

\[ f_\nu(x) \equiv \Phi(a, c, x)^2 - \Phi(a+\nu, c, x)\Phi(a-\nu, c, x). \]

We will make use of the following contiguous relation (see [10, p. 1013]):

\[ \Phi(\alpha + 1, \beta, x) - \Phi(\alpha, \beta, x) = \frac{x}{\beta} \Phi(\alpha + 1, \beta + 1, x). \]

Subtracting and adding a term to \( f_{\nu+1}(x) - f_\nu(x) \) and applying this contiguous relation, we have that
\[ f_{v+1}(x) - f_v(x) = \Phi(a + v, c, x)\Phi(a - v, c, x) - \Phi(a + v + 1, c, x)\Phi(a - v - 1, c, x) = \Phi(a - v, c, x)(\Phi(a + v, c, x) - \Phi(a + v + 1, c, x)) + \Phi(a + v + 1, c, x)(\Phi(a - v, c, x) - \Phi(a - v - 1, c, x)) = \Phi(a - v, c, x)\left(-\frac{x}{c}\right)\Phi(a + v + 1, c, x + 1) + \Phi(a + v + 1, c, x)\left(-\frac{x}{c}\right)\Phi(a - v + 1, c, x) = \frac{x}{c}g_v(x), \]

where

\[ g_v(x) = \Phi(a + v + 1, c, x)\Phi(a - v, c, x) - \Phi(a - v + 1, c, x)\Phi(a + v, c, x). \]

The Cauchy product reveals

\[ g_v(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(a + v + 1)_k(a - v)_{n-k}}{k!(n-k)!} \left( \frac{1}{(c)_k(c + 1)_{n-k}} - \frac{1}{(c)_{n-k}(c + 1)_{k}} \right) x^n. \]

where \( T_{n,k} \equiv \frac{(a + v + 1)_k(a - v)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_{k}}. \) If \( n \) is even, then

\[ \sum_{k=0}^{n/2-1} T_{n,k}(2k - n) = \sum_{k=0}^{n/2-1} T_{n,k}(2k - n) + \sum_{k=n/2+1}^{n} T_{n,k}(2k - n) = \sum_{k=0}^{n/2-1} T_{n,k}(2k - n) + \sum_{k=0}^{n/2-1} T_{n,n-k}(2(n - k) - n) = \sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n - 2k), \]

where \([\cdot] \) denotes the greatest integer function. Similarly, if \( n \) is odd, then

\[ \sum_{k=0}^{n} T_{n,k}(2k - n) = \sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n - 2k). \]

Therefore,

\[ f_{v+1}(x) - f_v(x) = \frac{x}{c}g_v(x) = \frac{x}{c^2} \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n - 2k)x^n. \]  \hspace{1cm} (6)

Simplifying, we find that

\[ T_{n,n-k} - T_{n,k} = \frac{(a + v + 1)_k(a - v)_{n-k} - (a + v + 1)_k(a - v)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_{k}} \]

\[ = \frac{(a + v + 1)_k(a - v)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_{k}} \left( (a + v + 1)_{n-k} - (a - v)_{n-k} \right) \]

\[ = \frac{(a + v + 1)_k(a - v)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_{k}} \left( (a + v + 1)_{n-k} - (a - v)_{n-k} \right) \]

\[ = \frac{(a + v + 1)_k(a - v)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_{k}} (h(a + v + 1) - h(a - v)). \]  \hspace{1cm} (7)

where \( h(\beta) \equiv \frac{\Gamma(\beta)}{\Gamma(\beta-k)}. \) For \( \beta > 0 \) and \( n - k > k \) (i.e., for \( [(n-1)/2] \geq k \)), the logarithmic derivative of \( h \) satisfies

\[ h'(\beta) = \Psi(\beta + n - k) - \Psi(\beta + k) > 0, \]

where \( \Psi \equiv \Gamma' / \Gamma \) is the digamma function. Hence, \( h \) is increasing under the conditions stated. This fact together with (6) and (7) yield
when \( a \geq v \geq 0 \), since \( x > 0 \) and \( c + 1 > 0 \), \( c \neq 0 \).

Thus, \((8)\) implies that 
\[
 f_{v+1}(x) - f_v(x) = \frac{x}{c^2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (T_{n,n-k} - T_{n,k})(n - 2k)x^k > 0
\]
when \( a \geq v \geq 0 \), since \( x > 0 \) and \( c + 1 > 0 \), \( c \neq 0 \).

Proof of Corollary 2. It follows from the proof of Theorem 1 that \((9)\) holds for \( x > 0 \) under the conditions that \( c + 1 > 0 \), \( c \neq 0 \). \(\square\)

Proof of Corollary 3. First suppose \( x \geq 0 \) and let \( a, b \geq v, v \in \mathbb{N} \). The first inequality in \((4)\) is a direct consequence of the fact that \( A((a+v)_n, (a-v)_n) = (a)_n \) for \( n = 0, 1 \) and
\[
 A((a+v)_n, (a-v)_n) > (a)_n \quad \text{for all } n \geq 2,
\]
which follows by induction. Thus,
\[
 A(\Phi(a+v, a+b, x), \Phi(a-v, a+b, x)) = \sum_{n=0}^{\infty} \frac{A((a+v)_n, (a-v)_n)x^n}{(a)_n n!} > \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(a+b)_n n!} = \Phi(a+b, x).
\]
For \( x \geq 0 \), the second inequality in \((4)\) follows by taking the square-root across \((1)\), which is allowed since the right-hand side of \((1)\) is nonnegative when \( a, b \geq v \).

Now suppose \( x < 0 \) with \( a, b \geq v \). Interchanging \( a \) and \( b \) in \((4)\), we have
\[
 A(\Phi(b+v, a+b, -x), \Phi(b-v, a+b, -x))
\]
\[
 > \Phi(b+a, b, -x) > G(\Phi(b+v, a+b, -x), \Phi(b-v, a+b, -x)).
\]
Kummer's transformation and the homogeneity of \( A \) and \( G \) yield
\[
e^{-x}A(\Phi(a-v, a+b, x), \Phi(a+v, a+b, x)) > e^{-x}G(\Phi(a-v, a+b, x), \Phi(a+v, a+b, x)).
\]
Thus, \((4)\) also holds for \( x < 0 \). \(\square\)

4. Concluding remarks

The proof of Theorem 1 can also be used to verify cases when the Turán-type inequality reverses. For example, if \( a < 0 \) and \( c + 1 < 0 \) with \( \lfloor a \rfloor = \lfloor c+1 \rfloor \), then
\[
 \Phi(a, c, x)^2 < \Phi(a+1, c, x)\Phi(a-1, c, x) \quad \text{for all } x > 0.
\]
To see this, take \( v = 0 \) in \((6)\) and then simplify to find that
\[
 \Phi(a, c, x)^2 - \Phi(a+1, c, x)\Phi(a-1, c, x) = \frac{x}{a^2 c^2} \sum_{n=1}^{\infty} \left( \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(a)_k (a)_n-k (n-2k)^2}{(c+1)_k (c+1)_n-k! (n-k)!} \right)x^k.
\]
The result follows by noting that \((a)_m\) and \((c+1)_m\) will have the same signs under the stated conditions (unless \((a)_m = 0\) for some \(m \geq 2\)). Hence
\[
\frac{1}{\alpha^2} \sum_{k=0}^{\left\lfloor \frac{(n-1)/2}{}\right\rfloor} \frac{(a)_k(a)_{n-k}(n-2k)^2}{(c+1)_k(c+1)_{n-k}k!(n-k)!} \leq 0
\]
for all \(n \in \mathbb{N}\). Moreover, the first nonzero term in the series in (10) simplifies to \(\frac{c^2}{c(c+1)} < 0\).

Finally, we note that the techniques of proof presented here can be used to obtain a result similar to (4) with \(\Phi = _1F_1\) replaced by the generalized hypergeometric function \(pF_q\), where
\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; x) \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!}.
\]

It can be shown that if \(p \leq q + 1, \alpha > 1, b_j > 0\) for \(i = 1, \ldots, q\), and \(a_i > b_i\) for \(i = 1, \ldots, p - 1\), then for all \(x > 0\) in the interval of convergence,
\[
A(pFq(\alpha + 1, a_1; b_1; x), pFq(\alpha - 1, a_1; b_1; x)) > pFq(\alpha, a_1; b_1; x) > G(pFq(\alpha + 1, a_1; b_1; x), pFq(\alpha - 1, a_1; b_1; x))
\]
(11)
where \(pFq(\alpha, a_1; b_1; x) \equiv pFq(\alpha, a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; x)\).

Of particular interest is the case that \(p = 2\) and \(q = 1\). In this case, inequality (11) completes the results of M.E.H. Ismail and A. Laforgia [11] and of Á. Baricz [3,5] regarding the Gaussian hypergeometric function \(2F_1\). See for example Theorems 2.13 and 2.14 in [11] and Theorem 2.17 in [3].

The first inequality in (11) follows as in Theorem 1. The second inequality in (11) follows by using a generalized version of (5) (see [14, Identity 30, p. 440]) to reveal that
\[
F(x) \equiv pFq(\alpha, a_1; b_1; x) - pFq(\alpha + 1, a_1; b_1; x) > x^2 \prod_{i=1}^{p-1} \left(\frac{(\alpha)_i}{(a_i)_i} \prod_{j=i}^{p-1} \frac{(a_i + 1)_j}{(b_j)_j} - \frac{(\alpha + 1)_i}{(a_i + 1)_i} \prod_{j=i}^{p-1} \frac{(a_i + 1)_j}{(b_j)_j} \right) \prod_{i=1}^{q} \frac{(b_i + 1)_i}{(b_i)_i} R_{n,k}(n - 2k),
\]
where
\[
R_{n,k} = \frac{\prod_{i=1}^{p} (b_i + n - k)}{\prod_{i=1}^{p} (a_i + n - k)} - \frac{\prod_{i=1}^{q} (b_i + k)}{\prod_{i=1}^{q} (a_i + k)} (n - 2k).
\]
For \(n - k > k\), the positivity of \(R_{n,k}\) (and hence \(F\)) follows when \(r \mapsto \frac{\prod_{i=1}^{r} (b_i + r)}{\prod_{i=1}^{r} (a_i + r)}\) is increasing, which is the case under the stated conditions on the \(a_i\)'s and \(b_i\)'s.

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