1. Consider the following vectors in $\mathbb{R}^5$: $\mathbf{x} = (2, 1, 1, 1, 4)^T$, and $\mathbf{y} = (3, 6, 1, 5, 7)^T$.
   a. Find $\langle \mathbf{x}, \mathbf{y} \rangle$, $||\mathbf{x}||^2$, $||\mathbf{y}||^2$, $\hat{\mathbf{y}} = p(\mathbf{y}|\mathbf{x})$, and $\mathbf{y} - \hat{\mathbf{y}}$. Show that $\mathbf{x} \perp (\mathbf{y} - \hat{\mathbf{y}})$, and $||\mathbf{y}||^2 = ||\hat{\mathbf{y}}||^2 + ||\mathbf{y} - \hat{\mathbf{y}}||^2$.
   b. Let $\mathbf{w} = (-1, 2, 4, -4, 0)^T$ and $\mathbf{z} = 3\mathbf{x} + 2\mathbf{w}$. Show that $\langle \mathbf{w}, \mathbf{x} \rangle = 0$ and that $||\mathbf{z}||^2 = 9||\mathbf{x}||^2 + 4||\mathbf{w}||^2$. (Why must this be true?)
   c. Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, $A_3 = \{5\}$. Find $p(\mathbf{y}|\mathbf{A}_i)$, $i = 1, 2, 3$.

2. Is projection a linear transformation in the sense that $p(c\mathbf{y}|\mathbf{x}) = cp(\mathbf{y}|\mathbf{x})$ for any real number $c$? Prove or disprove. What is the relationship between $p(\mathbf{y}|\mathbf{x})$ and $p(\mathbf{y}|c\mathbf{x})$ for $c \neq 0$?

3. Suppose $||\mathbf{x}||^2 > 0$. Use calculus to prove that $||\mathbf{y} - b\mathbf{x}||^2$ is minimum for $b = \mathbf{y}^T\mathbf{x}/||\mathbf{x}||^2$. (That is, provide a calculus-based proof of the Theorem on p. 22 of the notes as an alternative to the one provided there.)

4. Here we will consider subspaces of $\mathbb{R}^4$. Let $\mathbf{x}_1 = (1, 1, 1, 0)^T$, $\mathbf{x}_2 = (1, 1, 0, 0)^T$, $\mathbf{x}_3 = (1, 0, 0, 1)^T$ and $\mathbf{x}_4 = (10, 3, 0, 7)^T$. Let $V_2 = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2)$, $V_3 = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $V_4 = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$.
   a. Find matrices $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ whose column spaces are equal to $V_2$, $V_3$, $V_4$, respectively.
   b. Find the dimensions of $V_2$ and $V_3$ (i.e., the ranks of $\mathbf{A}_2$ and $\mathbf{A}_3$).
   c. Find bases for $V_2$ and $V_3$ that contain vectors with as many zeros as possible.
   d. Give a vector $\mathbf{z} \neq \mathbf{0}$ that is orthogonal to all vectors in $V_3$.
   e. $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{z}$ are linearly independent. Therefore, $\mathbf{x}_4$ is expressible in the form $\sum_{i=1}^{3} b_i \mathbf{x}_i + c\mathbf{z}$ (this is true because one can form at most $d$ linearly independent vectors in a vector space of dimension $d$). Show that $c = 0$ and hence that $\mathbf{x}_4 \in V_3$, by determining $\mathbf{x}_4^T\mathbf{z}$. What is $\dim(V_4)$ (the dimension of $V_4$)?
   f. Give a simple description of $V_3$ in words.

5. Consider $\mathbb{R}^6$, Euclidean 6-space, but let’s index a vector $\mathbf{y}$ in this space as $\mathbf{y} = (y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{31})^T$. Think of this indexing scheme as representing data $y_{ij}$ from 3 treatment groups, with 3, 2, and 1 replicates (indexed by $j$) in groups 1, 2, and 3, respectively (indexed by $i$). That is, $y_{ij}$ represents the $j$th response in the $i$th group where $i = 1, 2, 3$ and $j = 1, \ldots, n_i$, $n_1 = 3$, $n_2 = 2$, $n_3 = 1$. Let $a_i$ be the indicator for the $i$th treatment group (e.g., $a_1 = (1, 1, 1, 0, 0, 0)^T$, $a_2 = (0, 0, 0, 1, 1, 0)^T$, etc.). Let $V = \mathcal{L}(a_1, a_2, a_3)$.
   a. For $\mathbf{y} = (2, 1, 0, 7, 9, 3)^T$ find $\hat{\mathbf{y}} = p(\mathbf{y}|V)$, $\mathbf{y} - \hat{\mathbf{y}}$, $||\mathbf{y}||^2$, $||\hat{\mathbf{y}}||^2$, $||\mathbf{y} - \hat{\mathbf{y}}||^2$.
   b. Give a general non-matrix formula for $\hat{\mathbf{y}} = p(\mathbf{y}|V)$ for any $\mathbf{y}$.

6. Problem 2.23 from the textbook.

7. Problem 2.24