CHAPTER 6

DISTRIBUTIONS OF FUNCTIONS OF R.V.'S

Want to find the distribution (say the p.d.f.) of $X = g(Y_1, \dots, Y_n)$.

We will use 3 methods:

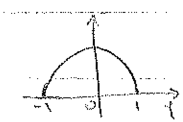
1. Method of distribution functions.
2. Method of transformations. (pdf's)
3. Method of moment generating functions.

} the 3 ways to characterize a distribution

1. METHOD OF DF's (§6.3)

Find the d.f. of X directly, $F_X(x) = P(X \le x)$. Then differentiate to get the p.d.f.

Ex. Suppose Y has density $f(y) = \begin{cases} \frac{3}{4}(1-y^2), & -1 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$



Find the density of

$$X = 2Y+3.$$

(what is the support of X?) $x \in (1,5)$

$$P(X \le x) = P(2Y+3 \le x) = P\left(Y \le \frac{x-3}{2}\right)$$

$$= \begin{cases} 0 & \text{if } \frac{x-3}{2} < -1 \iff x < 1 \\ \int_{-1}^{\frac{x-3}{2}} \frac{3}{4}(1-y^2)\,dy & \text{if } -1 \le \frac{x-3}{2} \le 1 \iff 1 \le x \le 5 \\ 1 & \text{if } \frac{x-3}{2} > 1 \iff x > 5 \end{cases}$$

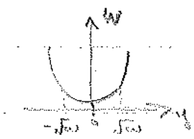
$$\int_{-1}^{\frac{x-3}{2}} \frac{3}{4}(1-y^2)\,dy = \left(\frac{3}{4}y - \frac{1}{4}y^3\right)\Big|_{y=-1}^{\frac{x-3}{2}} = \frac{3}{4}\left(\frac{x-3}{2}\right) - \frac{1}{4}\left(\frac{x-3}{2}\right)^3 + \frac{1}{2}$$

Differentiate

$$f_X(x) = \begin{cases} \frac{3}{8}\left[1 - \left(\frac{x-3}{2}\right)^2\right], & \text{if } 1 \le x \le 5, \\ 0, & \text{o/w.} \end{cases}$$

<u>Ex</u>. (ctd) Find the pdf of $W = Y^2$. What is the support of W? $w \in (0,1)$

$$F_W(w) = P(W \le w) = P(Y^2 \le w) = \begin{cases} 0 & \text{if } w < 0 \\ P(-w^{\frac{1}{2}} \le Y \le w^{\frac{1}{2}}) & \text{if } 0 \le w \le 1 \\ 1 & \text{if } w > 1 \end{cases}$$



For $0 \le w \le 1$,

$$P(-w^{\frac{1}{2}} \le Y \le w^{\frac{1}{2}}) = \int_{-w^{\frac{1}{2}}}^{w^{\frac{1}{2}}} \tfrac{3}{4}(1-y^2)\,dy = \left(\tfrac{3}{4}y - \tfrac{1}{4}y^3\right)\Big|_{y=-w^{\frac{1}{2}}}^{w^{\frac{1}{2}}}$$

$$= \left(\tfrac{3}{4}w^{\frac{1}{2}} - \tfrac{1}{4}w^{3/2}\right) - \left(-\tfrac{3}{4}w^{\frac{1}{2}} + \tfrac{1}{4}w^{3/2}\right) = \tfrac{3}{2}w^{1/2} - \tfrac{1}{2}w^{3/2}.$$

$$f_W(w) = \begin{cases} \tfrac{3}{4}\left(w^{-\frac{1}{2}} - w^{\frac{1}{2}}\right), & 0 < w < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

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* <u>Ex</u>. Let Y_1, Y_2 have jt pdf

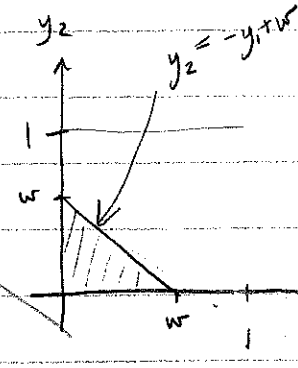
$$f(y_1, y_2) = 4y_1y_2, \quad 0 \le y_1 \le 1, \ 0 \le y_2 \le 1.$$

Find the density of $W = Y_1 + Y_2$. ← Think about the "support" of W.

$$F_W(w) = P(W \le w) = P(Y_1 + Y_2 \le w)$$

$F_W(w) = 0$ if $w < 0$ and $F_W(w) = 1$ if $w > 2$.

Otherwise $y_1 + y_2 \le w \Rightarrow y_2 \le -y_1 + w$.



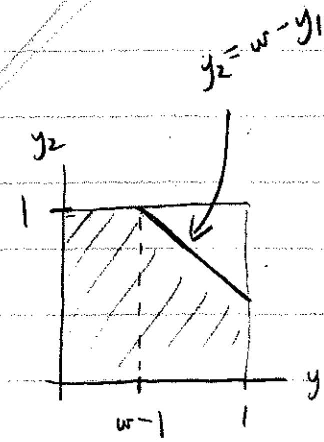
$0 \le w \le 1$

skip!

If $0 < w \le 1$,

$$F_W(w) = \int_0^w \int_0^{w-y_1} 4y_1y_2 \, dy_2 \, dy_1 = w^4/6 .$$

If $1 \le w \le 2$,



$$F_W(w) = 1 - \int_{w-1}^{1} \int_{w-y_1}^{1} 4y_1y_2 \, dy_2 \, dy_1$$

$$= 1 - \frac{8}{3}w + 2w^2 - \frac{1}{6}w^4$$

So

$$F_W(w) = \begin{cases} 0 & , \quad w < 0 \\ \frac{1}{6}w^4 & , \quad 0 \le w \le 1 \\ 1 - \frac{8}{3}w + 2w^2 - \frac{1}{6}w^4 & , \quad 1 \le w \le 2 \\ 1 & , \quad w > 2 \end{cases}$$

and thus

$$f_W(w) = \begin{cases} \frac{2}{3}w^3 & , \quad 0 \le w \le 1 \\ -\frac{8}{3} + 4w - \frac{2}{3}w^3 & , \quad 1 \le w \le 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Ex. Let Y_1, Y_2 have jt pdf

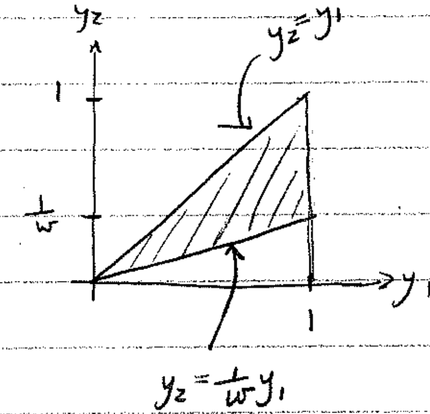
$$f(y_1,y_2) = \begin{cases} 3y_1 & , \quad 0 \le y_2 \le y_1 \le 1, \\ 0 & , \quad \text{elsewhere}. \end{cases}$$

Find the pdf of

$$W = Y_1 / Y_2 .$$

← Determine the "support" of W now. $1 \le w < \infty$

$$F_W(w) = P(W \le w) = P\left(\frac{Y_1}{Y_2} \le w\right).$$



$\frac{y_1}{y_2} \le w \iff y_2 \ge \frac{1}{w} y_1$

Note that $F_W(w) = 0$ if $\frac{1}{w} > 1 \iff w < 1$.

$w < 1$

For $\frac{1}{w} \le 1$, i.e., $w \ge 1$,

$$F_W(w) = P\left(\frac{Y_1}{Y_2} \le w\right) = P\left(Y_2 \ge \frac{1}{w} Y_1\right)$$

$$= \int_0^1 \int_{\frac{1}{w}y_1}^{y_1} 3y_1 \, dy_2 \, dy_1 = \int_0^1 3y_1\left(y_1 - \frac{1}{w} y_1\right) dy_1$$

$$= \int_0^1 3\left(1 - \frac{1}{w}\right) y_1^2 \, dy_1 = 1 - \frac{1}{w}.$$

$$F_W(w) = \begin{cases} 0 & , \quad w < 1 \\ 1 - \frac{1}{w} & , \quad w \ge 1 \end{cases}$$

$$f_W(w) = \begin{cases} \frac{1}{w^2} & , \quad \text{for } w \ge 1, \\ 0 & , \quad \text{o/w}. \end{cases}$$

Motivation: How to draw $y \in Y \sim \text{Exp}(\beta)$? (132)

Probability Integral Transformation

NOTE. Suppose $U \sim$ Uniform on $(0,1)$, i.e.,

$$f_U(u) = \begin{cases} 1, & 0 \le u \le 1, \\ 0, & \text{elsewhere}, \end{cases} \Rightarrow F_U(u) = \begin{cases} 0, & u < 0 \\ u, & 0 \le u \le 1 \\ 1, & u > 1 \end{cases}$$

Let F be a continuous distribution function with inverse F^{-1}, and define a random variable Y as

$$Y = F^{-1}(U). \qquad (\text{Note: } F(Y) = U)$$

Then

$$F_Y(y) = P(Y \le y) = P(F^{-1}(U) \le y)$$

$$= \underbrace{P(U \le F(y))}_{\text{“} F_U(F(y)) \text{”}} = F(y)$$

$\Rightarrow$ Y has distribution function F.

Explain use in random number generation. $\boxed{u \in U} \xrightarrow{F_Y^{-1}} \boxed{F_Y^{-1}(u) \in Y}$

Ex. Let $U \sim U(0,1)$. Suppose we want $Y \sim \text{Exp}(\beta)$, $\beta > 0$.

Exp(β) dist has density $f(y) = \frac{1}{\beta} e^{-y/\beta}$, $y > 0$

$$F(y) = \int_0^y \frac{1}{\beta} e^{-y/\beta} dy = 1 - e^{-y/\beta}, \quad y \ge 0. \quad \leftarrow \text{worry just about the } y \text{ in the support of } Y.$$

To find F^{-1}: $u = F(y) = 1 - e^{-y/\beta} \iff e^{-y/\beta} = 1 - u$

$$\iff -y/\beta = \ln(1-u) \iff y = -\beta \ln(1-u)$$

So take $Y = -\beta \ln(1 - U)$. Note that $1 - U \sim U(0,1)$ also, so could also take $Y = -\beta \ln(U)$

Ex (ctd). Check that for $y > 0$,

$$F_Y(y) = P(Y \le y) = P\left(-\beta \ln(1-U) \le y\right)$$

$$= P\left(\ln(1-U) \ge -\tfrac{y}{\beta}\right) = P\left(1-U \ge e^{-y/\beta}\right)$$

$$= P\left(U \le 1-e^{-y/\beta}\right) \overset{!}{=} 1-e^{-y/\beta}.$$

Ex. Could also do Pareto dist, which has $(\alpha > 0, \beta > 0)$

$$F(y) = 1 - \left(\frac{\beta}{y}\right)^{\alpha}, \quad \text{for } y \ge \beta.$$

worry just about y in the support of Y.

$$u = 1 - \left(\frac{\beta}{y}\right)^{\alpha} \iff \left(\frac{\beta}{y}\right)^{\alpha} = 1-u \iff \frac{\beta}{y} = (1-u)^{1/\alpha}$$

$$\iff y = \beta(1-u)^{-1/\alpha} \implies F^{-1}(u) = \beta(1-u)^{-1/\alpha}$$

So if $U \sim U(0,1)$, then $Y = \beta(1-U)^{-1/\alpha} \sim \text{Pareto}(\alpha, \beta)$.

So, to generate ~~a~~ random value y from $\text{Exp}(\beta)$, generate $u \sim U(0,1)$

$$\implies y = F^{-1}(u) = -\beta \ln(1-u).$$

~~is a random d~~

How to generate $u \in U(0,1)$?

[Pseudo-random # generators.]

Do Probs 6.3

2. Method of Transformations

(i) Univariate Transforms (§6.4)

<u>Theom</u> Let Y be a cts r.v. with pdf $f_Y(y)$.
Let $X = h(Y)$, where $h(y)$ is a monotone function (is either always increasing or always decreasing).

h(·) must be 1-1
↓ or ↑ ensures this

Then:

$$f_X(x) = \left| \frac{dh^{-1}(x)}{dx} \right| f_Y(h^{-1}(x)).$$

<u>Ex</u>: $f(y) = \frac{3}{4}(1-y^2)$, $-1 \le y \le 1$.

$X = 2Y + 3$ ← <u>NB</u>: $h(\cdot) \uparrow$

Here: $x = h(y) = 2y+3 \Rightarrow y = h^{-1}(x) = \frac{x-3}{2}$

Supp X: $h(-1) = 1 \le x \le h(1) = 5 \Rightarrow x \in (1,5)$.

$$\frac{dh^{-1}(x)}{dx} = \frac{d}{dx}\left(\frac{x-3}{2}\right) = \frac{1}{2}$$

$$\Rightarrow f_X(x) = \left| \frac{dh^{-1}(x)}{dx} \right| f_Y\left(\frac{x-3}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{3}{4}\left[1 - \left(\frac{x-3}{2}\right)^2\right]$$

$$= \frac{3}{8}\left[1 - \left(\frac{x-3}{2}\right)^2\right], \quad 1 \le x \le 5.$$

(Same as before ...)

<u>Ex.</u> (Exercise 6.68) If $Y \sim N(\mu, \sigma^2)$ and $W = e^Y$, the W is said to have a lognormal ~~equation~~ dist. Find the density of W.

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}, \quad -\infty < y < \infty.$$

$$w = e^y, \quad y = \ln(w), \quad \frac{dy}{dw} = \frac{1}{w}, \quad w > 0.$$

(↑ so ok)

$$f_W(w) = f_Y(y)\left|\frac{dy}{dw}\right| = \frac{1}{w\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(w)-\mu}{\sigma}\right)^2}, \quad w > 0.$$

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Ex. Again suppose Y has pdf $f_Y(y) = \frac{3}{4}(1-y^2)$, $-1 \le y \le 1$, and suppose

$$W = 2 - \tfrac{1}{2}Y.$$

$$w = h(y) = 2 - \tfrac{1}{2}y \qquad y = h^{-1}(w) = 4 - 2w$$

$$\left|\frac{dy}{dw}\right| = |-2| = 2$$

$$-1 \le 4 - 2w \le 1 \iff -5 \le -2w \le -3 \qquad \tfrac{3}{2} \le w \le \tfrac{5}{2}.$$

So

$$f_W(w) = f_Y(y)\left|\frac{dy}{dw}\right|$$

$$= \tfrac{3}{4}\left[1-(4-2w)^2\right]\cdot 2$$

$$= \tfrac{3}{2}\left[1-(4-2w)^2\right], \qquad \tfrac{3}{2} \le w \le \tfrac{5}{2}.$$

STRESS: $h(y)$ must be a 1-1 function. Otherwise use a different method. Method of transformations can be adapted, but more complicated

Ex. Suppose Y has p.d.f.

$$f_Y(y) = e^{-y}, \qquad y > 0. \qquad (Y \sim \text{Exp}(1)).$$

Let

$$W = \tfrac{1}{Y}. \qquad \text{Supp } W = (0,\infty).$$

Note that $w = h(y) = \frac{1}{y}$ is decreasing for $y > 0$, and

$$y = h^{-1}(w) = \tfrac{1}{w} \qquad \frac{dy}{dw} = \frac{-1}{w^2}.$$

Thus

$$f_W(w) = f_Y(y)\left|\frac{dy}{dw}\right| = e^{-\frac{1}{w}}\cdot\left|\frac{-1}{w^2}\right| = \frac{1}{w^2}e^{-w^{-1}}, \qquad w > 0.$$

Do Probs 6.4

(ii) Bivariate Transforms (§6.6)

$Y_1, Y_2 \sim f(y_1, y_2)$, and $\begin{cases} x_1 = h_1(y_1, y_2) \\ x_2 = h_2(y_1, y_2) \end{cases}$

is a 1-1 transform from $(y_1, y_2) \longmapsto (x_1, x_2)$,
with inverse: $y_1 = h_1^{-1}(x_1, x_2)$, $y_2 = h_2^{-1}(x_1, x_2)$.

If $J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial x_1} & \frac{\partial h_1^{-1}}{\partial x_2} \\ \frac{\partial h_2^{-1}}{\partial x_1} & \frac{\partial h_2^{-1}}{\partial x_2} \end{vmatrix}$, w/ cts partial derivatives,

then:

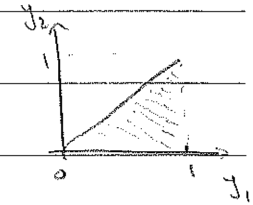
$$\boxed{f_{X_1, X_2}(x_1, x_2) = |J| \, f_{Y_1, Y_2}\Big(\underset{\substack{\uparrow \\ y_1}}{h_1^{-1}(x_1, x_2)}, \underset{\substack{\uparrow \\ y_2}}{h_2^{-1}(x_1, x_2)}\Big).}$$

generalizes to n dimensions ...

Often used to find just marginal dist of X_1, in which case we usually choose $X_2 = Y_2$
(simple, but caution that overall transform should still be 1-1.)

Ex. Let Y_1, Y_2 have jt. p.d.f.

$$f(y_1, y_2) = \begin{cases} 3y_1 , & 0 \le y_2 \le y_1 \le 1 \\ 0 , & o/w \end{cases}$$



Find the density of $W = Y_1/Y_2$.

Solution. Let $W_1 = W = Y_1/Y_2$ Note that $W \ge 1$.

$W_2 = Y_2$

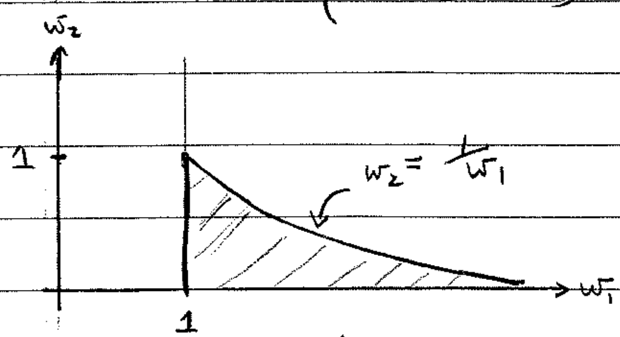
$$\begin{vmatrix} \frac{\partial y_1}{\partial w_1} & \frac{\partial y_1}{\partial w_2} \\ \frac{\partial y_2}{\partial w_1} & \frac{\partial y_2}{\partial w_2} \end{vmatrix}$$

$$\begin{matrix} w_1 = y_1/y_2 \\ w_2 = y_2 \end{matrix} \Rightarrow \begin{matrix} y_1 = w_1 w_2 \\ y_2 = w_2 \end{matrix} \Rightarrow J = \begin{vmatrix} w_2 & w_1 \\ 0 & 1 \end{vmatrix} = w_2$$

$$f_{W_1,W_2}(w_1, w_2) = f_{Y_1,Y_2}(y_1, y_2)\,|J| = \begin{cases} 3w_1 w_2 \,|w_2| , & 0 \le w_2 \le w_1 w_2 \le 1 \\ 0 , & o/w \end{cases}$$

$$= \begin{cases} 3w_1 w_2^2 , & w_1 \ge 1,\ 0 \le w_2 \le \frac{1}{w_1} \quad \text{or} \quad 0 \le w_2 \le 1,\ 1 \le w_1 \le \frac{1}{w_2} \\ 0 , & o/w \end{cases}$$

(2 equiv. ways of describing region)



For $w_1 \ge 1$ $\quad f_{W_1}(w_1) = \int_0^{1/w_1} 3w_1 w_2^2 \, dw_2 = 3w_1 \int_0 w_2^2 \, dw_2 = \cancel{3}w_1 \, \frac{1}{\cancel{3}} w_2^3 \Big|_0^{1/w_1}$

$$= w_1 \cdot \frac{1}{w_1^3} = \frac{1}{w_1^2} , \quad w_1 > 1$$

* Ex. Let Y_1, Y_2 have density

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 4y_1y_2 & , 0 \le y_1 \le 1, \ 0 \le y_2 \le 1 \\ 0 & , \text{o/w} \end{cases}$$

Find the density of $W = Y_1 + Y_2$.

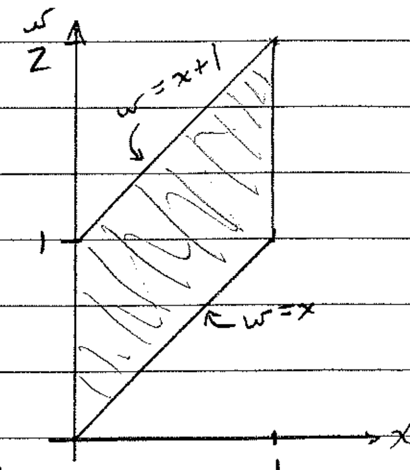
Sol'n. Let $X = Y_2$.

$$\begin{cases} w = y_1 + y_2 \\ x = y_2 \end{cases} \Rightarrow \begin{cases} y_1 = w - x \\ y_2 = x \end{cases} \Rightarrow \begin{cases} \frac{\partial y_1}{\partial w} = 1 & \frac{\partial y_1}{\partial x} = -1 \\ \frac{\partial y_2}{\partial w} = 0 & \frac{\partial y_2}{\partial x} = 1 \end{cases}$$

$$\Rightarrow J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{W,X}(w,x) = f_{Y_1,Y_2}(y_1,y_2)\,|J| = \begin{cases} 4(w-x)x \cdot |1| & \overbrace{0 \le w - x \le 1, \ 0 \le x \le 1}^{x \le w \le x+1, \ 0 \le x \le 1} \\ 0 & , \text{o/w} \end{cases}$$

Now: $f_W(w) = \int_{\mathbb{R}} f_{W,X}(w,x)\,dx \qquad 0 \le x \le 1, \ x \le w \le x+1$



For $0 \le w \le 1$, $0 \le x \le w$, so

$$f_W(w) = \int_0^w 4(w-x)x\,dx$$

$$= \int_0^w (4wx - 4x^2)\,dx = \left(2wx^2 - \tfrac{4}{3}x^3\right)\Big|_0^w$$

$$= 2w^3 - \tfrac{4}{3}w^3 = \tfrac{2}{3}w^3$$

For $1 \le w \le 2$, $w - 1 \le x \le 1$, so

$$f_W(w) = \int_{w-1}^1 4(w-x)x\,dx = \left(2wx^2 - \tfrac{4}{3}x^3\right)\Big|_{x=w-1}^1$$

$$= -\tfrac{2}{3}w^3 + 4w - \tfrac{8}{3}$$

$$\Rightarrow f_W(w) = \begin{cases} \frac{2}{3}w^3 & , 0 \le w \le 1 \\ -\frac{2}{3}w^3 + 4w - \frac{8}{3} & , 1 \le w \le 2 \\ 0 & , \text{o/w} \end{cases}$$

Ex Y_1, Y_2 indep exponentials w/ mean $\beta > 0$. Find density of $U_1 = \frac{Y_1}{Y_1 + Y_2}$, and of $U_2 = Y_1 + Y_2$.

$$f_1(y_1) = \frac{1}{\beta} \exp(-y_1/\beta), \quad y_1 > 0 \quad \text{and} \quad f_2(y_2) = \frac{1}{\beta} \exp(-y_2/\beta), \quad y_2 > 0.$$

$$\Rightarrow f(y_1, y_2) = \frac{1}{\beta^2} \exp\left\{-\frac{1}{\beta}(y_1 + y_2)\right\}, \quad y_1 > 0, \ y_2 > 0.$$

$$\left. u_1 = \frac{y_1}{y_1 + y_2}, \quad u_2 = y_1 + y_2 \right\} \Rightarrow \begin{array}{ll} y_1 = u_1 u_2, & y_2 = u_2(1 - u_1) \\ \doteq h_1^{-1}(u_1, u_2), & = h_2^{-1}(u_1, u_2) \end{array}$$

$$\Rightarrow J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} u_2 & u_1 \\ -u_2 & 1 - u_1 \end{vmatrix} = u_2$$

$$\Rightarrow f_{U_1, U_2}(u_1, u_2) = \begin{cases} |u_2| \ \frac{1}{\beta^2} \exp\left\{-\frac{1}{\beta} u_2\right\}, & u_1 u_2 > 0, \ u_2(1 - u_1) > 0. \\ 0, & \text{o/w} \end{cases}$$

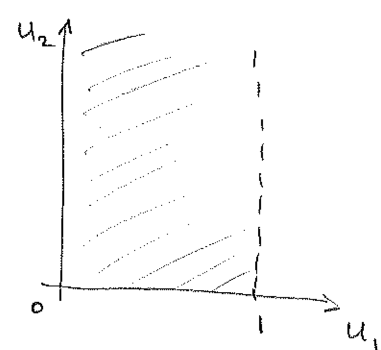
Since $y_1, y_2 > 0$, $0 < u_1 = \frac{y_1}{y_1 + y_2} < 1$, and $u_2 > 0$

$$\Rightarrow f_{U_1, U_2}(u_1, u_2) = \left. \begin{cases} \frac{u_2}{\beta^2} e^{-u_2/\beta}, & 0 < u_1 < 1, \ u_2 > 0 \\ 0, & \text{o/w} \end{cases} \right\} \begin{array}{l} U_1 \ \& \ U_2 \\ \text{indep.} \end{array}$$

$$f_{U_1}(u_1) = \int_0^\infty \frac{u_2}{\beta^2} e^{-u_2/\beta} \, du_2 = \int_0^\infty \frac{\beta^2}{\beta^2} x e^{-x} \, dx$$

$x = u_2/\beta$

$$= \underbrace{\Gamma(2)}_{1.}, \quad 0 < u_1 < 1 \quad \sim \text{Unif}(0,1).$$



$$f_{U_2}(u_2) = \int_0^1 \frac{u_2}{\beta^2} e^{-u_2/\beta} \, du_1 = \frac{1}{\beta^2} \cdot u_2 \, e^{-u_2/\beta}, \quad u_2 > 0.$$

$$= \frac{1}{\Gamma(2) \beta^2} u_2^{2-1} e^{-u_2/\beta} \quad \sim \text{Gamma}(\alpha = 2, \beta).$$

Do probs §6.6

SEC 6.5 METHOD OF MOMENT GENERATING FCNS

Thm. Suppose that X and Y are two distributions whose m.g.f.'s "exist" (finite in an interval containing the origin) and are denoted $m_X(t)$ and $m_Y(t)$. If $m_X(t) = m_Y(t)$ $\forall t$, then X and Y have the same ~~mgf~~ probability distribution.

The method here is, given a r.v. Y and $U = h(Y)$, we try to find $m_U(t) = E(e^{tU}) = E(e^{th(Y)})$. Then if we recognize the mgf of U as belonging to some known distribution we know the dist of U.

Useful results:

Thm. If Y has mgf $m_Y(t)$, then $W = aY+b$ has mgf

$$m_W(t) = e^{bt} m_Y(at).$$

Ex: Let $Y \sim \text{Gamma}(\alpha, \beta)$, and let $W = cY$, $c>0$.

Then, $m_W(t) = m_Y(ct) = (1-c\beta t)^{-\alpha} \Rightarrow W \sim \text{Gamma}(\alpha, c\beta)$

Thm If $Y_1, Y_2, \ldots, Y_n$ are independent r.v.'s with m.g.f.'s $m_{Y_1}(t), m_{Y_2}(t), \ldots, m_{Y_n}(t)$, resp., and if

$$U = Y_1 + Y_2 + \cdots + Y_n,$$

then

$$m_U(t) = m_{Y_1}(t) \cdot m_{Y_2}(t) \cdots m_{Y_n}(t).$$

Pf (easy).

Ex. $Y_i \sim \text{Poisson}(\lambda_i)$, $Y_1, \dots, Y_n$ indep, $U = Y_1 + \cdots + Y_n$.

$$m_{Y_i}(t) = e^{\lambda_i(e^t - 1)}.$$

$$m_U(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} \cdots e^{\lambda_n(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)} \leftarrow \text{mgf of Poisson}(\lambda_1+\cdots+\lambda_n).$$

So U has a Poisson dist with parameter $\sum_{i=1}^n \lambda_i$.

Ex. $Y_i \sim \text{Gamma}(\alpha_i, \beta)$, $i = 1, \dots, n$, $Y_1, \dots, Y_n$ independent.

$U = \sum_{i=1}^n Y_i$ $\qquad m_{Y_i}(t) = (1-\beta t)^{-\alpha_i}$.

$$m_U(t) = \prod_{i=1}^n m_{Y_i}(t) = \prod_{i=1}^n (1-\beta t)^{-\alpha_i}$$

$$= (1-\beta t)^{-\alpha_1}(1-\beta t)^{-\alpha_2} \cdots (1-\beta t)^{-\alpha_n}$$

$$= (1-\beta t)^{-(\alpha_1+\alpha_2+\cdots+\alpha_n)} = (1-\beta t)^{-\sum_{i=1}^n \alpha_i}.$$

So $U \sim \text{Gamma}\left(\alpha = \sum_{i=1}^n \alpha_i, \beta\right)$.

Must have same ~~shape~~ scale parameter β for all Y_i to do this.

Ex. Let $Y \sim N(\mu, \sigma^2)$. Recall: $m_Y(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$.

Consider: $X = aY + b$

Then:
$$m_X(t) = e^{bt} m_Y(at)$$
$$= e^{bt} e^{at\mu + \frac{1}{2}(at)^2\sigma^2}$$
$$= \exp\left\{ t(a\mu + b) + \tfrac{1}{2} t^2 (a\sigma)^2 \right\}$$

$$\Rightarrow \quad X \sim N(a\mu + b, a^2\sigma^2).$$

In part., if $a = 1/\sigma$ & $b = -\mu/\sigma$
$$X = \frac{Y - \mu}{\sigma} \sim N(0,1).$$

Ex: $Y_1, \dots, Y_n \sim$ indep, $Y_i \sim N(\mu_i, \sigma_i^2)$.

Consider: $U = a_1 Y_1 + \dots + a_n Y_n$.

Then, since $a_i Y_i \sim N(a_i \mu_i, a_i^2 \sigma_i^2)$

$$m_U(t) = \prod_{i=1}^{n} m_{Y_i}(a_i t)$$
$$= \prod_{1}^{n} e^{a_i t \mu_i + \frac{1}{2} a_i^2 t^2 \sigma_i^2}$$
$$= e^{t a_1 \mu_1 + \frac{1}{2} t^2 a_1^2 \sigma_1^2} \cdots e^{t a_n \mu_n + \frac{1}{2} t^2 a_n^2 \sigma_n^2}$$
$$= \exp\left\{ t(a_1\mu_1 + \dots + a_n\mu_n) + \tfrac{1}{2} t^2 (a_1^2\sigma_1^2 + \dots + a_n^2\sigma_n^2) \right\}$$

$$\Rightarrow \quad U \sim N\left(\sum_{1}^{n} a_i \mu_i, \sum_{1}^{n} a_i^2 \sigma_i^2 \right)$$

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}(y-\mu)^2/\sigma^2}$$

Ex. Let $Z \sim N(0,1)$, ~~$Y = Z$~~ $X = Z^2$.

$$m_X(t) = E\left(e^{tX}\right) = E\left(e^{tZ^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-\frac{1}{2}z^2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2(1-2t)}\, dz$$

Looks like $N\left(0, \frac{1}{1-2t}\right)$

Assume $t < \frac{1}{2}$

$1-2t>0 \Rightarrow$

$$= (1-2t)^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}\,(1-2t)^{-\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2/\left[(1-2t)^{-1}\right]}\, dz$$

$$= (1-2t)^{-\frac{1}{2}}, \quad t < 1/2 \longleftarrow \text{mgf of Gamma}\left(\alpha = \tfrac{1}{2}, \beta = 2\right)$$

$$\| \\ \chi^2 \text{ with } 1 \text{ df.}$$

[Recall the χ^2 with ν df is Gamma$\left(\alpha = \frac{\nu}{2}, \beta = 2\right)$.]

So the square of a standard normal r.v. has a χ^2 dist.

Thm 6.4 Ex. Let $Y_1, \dots, Y_n$ be ind with $Y_i \sim N(\mu_i, \sigma_i^2)$, let

$$Z_i = \left(\frac{Y_i - \mu_i}{\sigma_i}\right), \text{ and let } X = \sum_{i=1}^{n} Z_i^2.$$

Now by our earlier example, the Z_i are indep $N(0,1)$, and thus the Z_i^2 are indep. χ_1^2

<u>Theom</u>. Let $Y_1, \dots, Y_n \sim$ indep, $Y_i \sim N(\mu_i, \sigma_i^2)$.

Let: $\begin{cases} Z_i = \dfrac{Y_i - \mu_i}{\sigma_i} \\ X = \sum_1^n Z_i^2 \end{cases}$

Then $X \sim \chi^2_n$. (χ^2 with n df.)

<u>Proof</u>

From earlier we know:

$Z_i \sim N(0,1)$

$Z_i^2 \sim \chi^2_1$.

So $Z_1^2, \dots, Z_n^2 \sim$ indep with

$Z_i^2 \sim \chi^2_1 = \text{Gamma}(\alpha = \frac{1}{2}, \beta = 2)$

Since $Y_1, \dots, Y_n \sim$ indep, $Y_i \sim \text{Gamma}(\alpha_i, \beta)$

$\Rightarrow \sum_1^n Y_i \sim \text{Gamma}(\sum_1^n \alpha_i, \beta)$

we have:

$$X = \sum_1^n Z_i^2 = \sum_1^n Y_i, \quad Y_i \sim \text{Gamma}(\alpha_i = \tfrac{1}{2}, \beta = 2)$$

$$\Rightarrow X \sim \text{Gamma}(\sum_1^n \alpha_i, \beta) \sim \text{Gamma}(\sum_1^n \tfrac{1}{2}, 2)$$

$$\sim \text{Gamma}(\tfrac{n}{2}, 2)$$

$$\sim \chi^2_n.$$

RECALL: $Y \sim \text{Gamma}(\alpha, \beta) \Rightarrow W = cY \sim \text{Gamma}(\alpha, c\beta)$, $c > 0$.

So choosing $c = \frac{2}{\beta}$

$$\Rightarrow W = \frac{2}{\beta} Y \sim \text{Gamma}\left(\alpha, \frac{2}{\beta}\beta\right) \sim \text{Gamma}(\alpha, 2)$$
$$\sim \text{Gamma}\left(\frac{2\alpha}{2}, 2\right)$$
$$\sim \chi^2_{2\alpha}$$

$$\boxed{\therefore\ Y \sim \text{Gamma}(\alpha, \beta) \Rightarrow \frac{2}{\beta} Y \sim \chi^2_{2\alpha}.}$$

Ex. Suppose $Y \sim \text{Gamma}(10, 3)$.
Find $y_0 \ni$

$$P(Y > y_0) = .05.$$

95th % of Y

Let $W = \frac{2}{3}Y$. Then

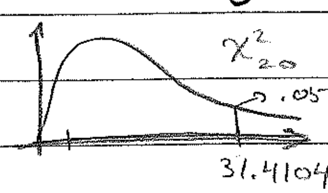
$$W \sim \text{Gamma}\left(\alpha = 10 = \frac{20}{2},\ \beta = \frac{2}{3} \cdot 3 = 2\right) = \chi^2_{20}$$

$$.05 = P(Y > y_0) = P\left(\frac{2}{3}Y > \frac{2}{3}y_0\right) = P(W > w_0)$$

where $w_0 = \frac{2}{3}y_0$

From χ^2-table, (Table 6)

$$w_0 = 31.4104$$



$$\Rightarrow \frac{2}{3}y_0 = 31.4104 \Rightarrow y_0 = \left(\frac{3}{2}\right)(31.4104) = 47.1156$$

Do probs 6.5

Sec 6.7. Order Statistics

Let $Y_1, Y_2, \ldots, Y_n$ denote independent, continuous random variables with cdf $F(y)$ and density $f(y)$. Denote the ordered random variables Y_i by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$, where

$$Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}.$$

So

$$Y_{(1)} = \min(Y_1, \ldots, Y_n) \text{ and } Y_{(n)} = \max(Y_1, \ldots, Y_n).$$

~~Give an example~~

Ex. ① Highest score on an exam in a class of n students $(Y_{(n)})$.

~~② First time that an~~

② Earliest failure time among n identical electronic components $(Y_{(1)})$.

③ Median height among n people $(Y_{(\frac{n+1}{2})}$, n ~~even~~ odd$)$

Consider the dist. of the maximum.

e.g. prob max score on final ≤ y

$$P(Y_{(n)} \le y) = P(Y_1 \le y, \ldots, Y_n \le y)$$

$$= P(Y_1 \le y) \cdots P(Y_n \le y) = [F(y)]^n$$

Letting G_n and g_n denote the cdf and pdf of $Y_{(n)}$, we thus have

$$G_n(y) = [F(y)]^n$$

and

$$\boxed{g_n(y) = \frac{d}{dy}\left\{[F(y)]^n\right\} = n[F(y)]^{n-1} f(y).}$$

Similarly ~~of~~ for $Y_{(1)}$ we have

$$P(Y_{(1)} \le y) = 1 - P(Y_{(1)} > y) = 1 - P(Y_1 > y, \dots, Y_n > y)$$

$$= 1 - P(Y_1 > y) \cdots P(Y_n > y) = 1 - [1 - F(y)]^n,$$

so that if G_1 and g_1 are the cdf and pdf of $Y_{(1)}$, resp., then

$$G_1(y) = 1 - [1 - F(y)]^n$$

and

$$\boxed{g_1(y) = -n[1 - F(y)]^{n-1}(-f(y)) = n[1 - F(y)]^{n-1} f(y).}$$

Now let $n = 2$ and consider the jt dist of $Y_{(1)}$ and $Y_{(2)}$.

Note that for $y_1 \le y_2$,

$$G_{12}(y_1, y_2) = P(Y_{(1)} \le y_1, Y_{(2)} \le y_2) = P(\{Y_1 \le y_1, Y_2 \le y_2\} \cup \{Y_2 \le y_1, Y_1 \le y_2\})$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= P(Y_1 \le y_1, Y_2 \le y_2) + P(Y_2 \le y_1, Y_1 \le y_2)$$

$$- P(Y_1 \le y_1, Y_2 \le y_1) \qquad \because y_1 \le y_2$$

$$\cancel{= 2F(y_1, y_2)} = 2F(y_1)F(y_2) - [F(y_1)]^2$$

$$g_{12}(y_1, y_2) = \cancel{\frac{\partial^2}{\partial y_1 \partial y_2} G(y_1, y_2) = 2F(y_1)f(y_2) - 2f(y_1)}$$

$$= \frac{\partial^2}{\partial y_1 \partial y_2} G_{12}(y_1, y_2) = \frac{\partial}{\partial y_1}\left[2F(y_1)f(y_2)\right]$$

$$= 2f(y_1)f(y_2), \qquad y_1 \le y_2.$$

Note: this says $Y_{(1)}$ & $Y_{(2)}$ dep.

Theom

$Y_1, \dots, Y_n \sim$ iid $f(y)$, $F(y)$, continuous.
order stats: $Y_{(1)} \le \dots \le Y_{(n)}$. Then:

(1) If $G_k(y)$ & $g_k(y)$ denotes cdf & pdf of $Y_{(k)}$:

$$g_k(y) = n\binom{n-1}{k-1} F^{k-1}(y)\left[1-F(y)\right]^{n-k} f(y), \quad -\infty < y < \infty$$

(2) Joint density of $(Y_{(j)}, Y_{(k)})$; $j < k$:

$$g_{j,k}(y_j, y_k) = \frac{n!}{(j-1)!(k-1-j)!(n-k)!} F^{j-1}(y_j)\left[F(y_k) - F(y_j)\right]^{k-1-j}$$

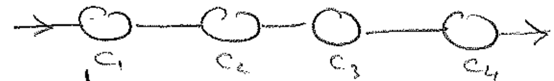
$$\times \left[1 - F(y_k)\right]^{n-k} f(y_j) f(y_k), \quad -\infty \le y_j \le y_k < \infty .$$

(3) Joint density of $(Y_{(1)}, \dots, Y_{(n)})$:

$$g_{1,\dots,n}(y_1, \dots, y_n) = n!\, f(y_1) \cdots f(y_n), \quad y_1 \le y_2 \le \dots \le y_n .$$

Note : ~~Orde~~ Although underlying $Y_1, \dots, Y_n$ are indep, the order stats $Y_{(1)}, \dots, Y_{(n)}$ are dependent!

Ex. A system consists of 4 identical electronic components in <u>series</u>.



The lifetimes of the components are indep ~~w~~ exponentially dist with mean **2**. Find the dist of the system's lifetime.

Let Y_i = lifetime of i^{th} component, $i = 1, \dots, 4$.

$$f(y) = \frac{1}{2} e^{-y/2}, \quad y > 0$$

$$F(y) = 1 - e^{-y/2}, \quad y > 0.$$

$Y_{(1)}$ is the system lifetime (~~and~~ 1^{st} failure time).

$$g_1(y) = 4\left[1 - F(y)\right]^{4-1} f(y) = 4\left[e^{-y/2}\right]^3 \frac{1}{2} e^{-y/2}$$

$$= 2e^{-2y}, \quad y > 0.$$

Note: $Y_{(1)} \sim \text{Exp}(\beta = 1/2)$.

In general, if $Y_1, \dots, Y_n \sim$ iid $\text{Exp}(\beta)$, then $Y_{(1)}$ has pdf:

$$g_1(y) = n\left[1 - F(y)\right]^{n-1} f(y)$$

$$= n\left(e^{-y/\beta}\right)^{n-1} \frac{1}{\beta} e^{-y/\beta}$$

$$= \frac{n}{\beta} e^{-ny/\beta}$$

$$= \begin{cases} \dfrac{n}{\beta} e^{-y/(\beta/n)}, & y > 0 \\ 0, & \text{o/w} \end{cases}$$

$\Rightarrow Y_{(1)} \sim \text{Exp}(\beta/n)$

$\Rightarrow \mathbb{E}\, Y_{(1)} = \frac{1}{n}\beta = \frac{1}{n}\mathbb{E}(Y_1)$.

Ex: $Y_1, \dots, Y_n \sim$ iid Unif$(0,1)$.

(a) Get pdf of $Y_{(k)}$, $1 \le k \le n$.

Know: $F(y) = \begin{cases} 0, & y<0 \\ y, & 0 \le y \le 1 \\ 1, & y>1 \end{cases}$, $f(y) = 1$, $0 \le y \le 1$.

$$\therefore\ g_k(y) = n\binom{n-1}{k-1} F^{k-1}(y)\left[1-F(y)\right]^{n-k} f(y), \quad 0 \le y \le 1$$

$$= n\binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}, \quad 0 \le y \le 1$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\, y^{k-1}(1-y)^{n-k}, \quad 0 \le y \le 1$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}\, y^{k-1}(1-y)^{n-k}, \quad 0 \le y \le 1.$$

Hence: $Y_{(k)} \sim \text{Beta}(\alpha = k, \beta = n-k+1)$

[Mention Claudio Lotto ; order stats from discrete unif]

(b) Get mean & variance of $Y_{(k)}$.

$$\mathbb{E}(Y_{(k)}) = \frac{k}{n+1}, \quad \text{Var}(Y_{(k)}) = \frac{k(n-k+1)}{(n+1)^2(n+2)}$$

(c) Compute mean distance between successive order stats.

$$\mathbb{E}\left[Y_{(k)} - Y_{(k-1)}\right] = \mathbb{E}\,Y_{(k)} - \mathbb{E}\,Y_{(k-1)}$$

$$= \frac{k}{n+1} - \frac{k-1}{n+1} = \frac{1}{n+1}$$

$\Rightarrow$ successive order stats (from a Unif) are on average equally spaced.

Do probs 6.7 . END OF COURSE (after next page #152)

(d) Compute $\text{Corr}(Y_{(k)}, Y_{(k-1)})$, $\quad 2 \le k \le n$.

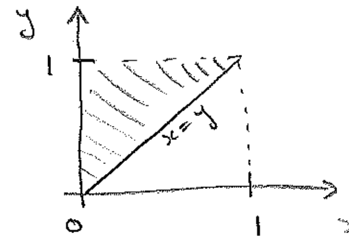
Let $x = y_{(k-1)}$, $y = y_{(k)}$, $x \le y$, and note that:

$$f(x,y) = \frac{n!}{(k-2)!\,0!\,(n-k)!} F^{k-2}(x)\left[F(y)-F(x)\right]^0\left[1-F(y)\right]^{n-k} f(x) f(y)$$

$$= c_{n,k}\, x^{k-2}(1-y)^{n-k}, \quad 0 \le x \le y \le 1, \quad c_{n,k} = k(k-1)C^n_k.$$

Now:

$$\mathbb{E}(XY) = \iint_{\{(x,y):\, 0\le x\le y\le 1\}} x y f(x,y)\, dx\, dy$$



$$= \int_0^1 \int_0^y c_{n,k}\, y(1-y)^{n-k} x^{k-1}\, dx\, dy$$

$$= \frac{c_{n,k}}{k} \underbrace{\int_0^1 y^{k+1}(1-y)^{n-k}\, dy}_{\text{kernel of Beta}(\alpha = k+2,\ \beta = n-k+1)}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma(k+2)\Gamma(n-k+1)}{\Gamma(n+3)} = \frac{(k+1)!\,(n-k)!}{(n+2)!}$$

$$\Rightarrow \mathbb{E}(XY) = \frac{\cancel{k}(k-1)}{\cancel{k}} \cdot \frac{\cancel{n!}}{\cancel{k!\,(n-k)!}} \cdot \frac{(k+1)\cancel{k!\,(n-k)!}}{\cancel{n!}\,(n+1)(n+2)} = \frac{(k+1)(k-1)}{(n+1)(n+2)}$$

$$\Rightarrow C(X,Y) = \mathbb{E}(XY) - \mathbb{E}X\cdot\mathbb{E}Y = \frac{(k+1)(k-1)}{(n+1)(n+2)} - \frac{k(k-1)}{(n+1)^2}$$

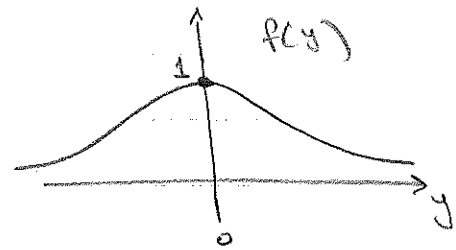
$$= \frac{(k-1)(n+1-k)}{(n+1)^2(n+2)}$$

Finally:
$$\rho = \frac{C(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{(k-1)(n+1-k)}{(n+1)^2(n+2)} \sqrt{\frac{(n+1)^2(n+2)}{k(n+1-k)} \cdot \frac{(n+1)^2(n+2)}{(k-1)(n+2-k)}}$$

$$= \sqrt{\frac{(k-1)(n+1-k)}{k(n+2-k)}} \approx 1 \text{ for large } n$$

Ex: $n=10$, $k=2$: $\text{Corr}(Y_{(1)}, Y_{(2)}) = 0.67$

Cauchy Distribution

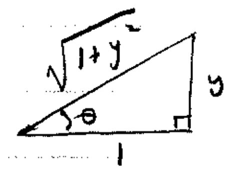


Let Y have p.d.f.

$$f(y) = k \cdot \frac{1}{1+y^2}, \quad -\infty < y < \infty.$$

What is k? Recall from calculus that

$$\int \frac{1}{1+y^2}\, dy = \arctan(y) + C$$



Why?

$$\int \frac{1}{1+y^2} dy = \int \cos^2(\theta) \cdot \sec^2(\theta)\, d\theta$$

$$= \int d\theta = \theta + C$$

$$= \arctan(y) + C$$

$\theta = \arctan y$

$y = \tan(\theta)$

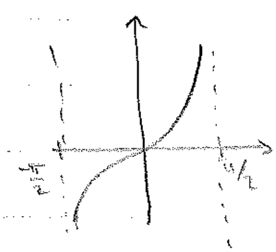
$\frac{1}{1+y^2} = \cos^2(\theta)$

$dy = \frac{1}{\cos^2\theta} d\theta = \sec^2\theta\, d\theta$

Now,

$$1 = \int_{-\infty}^{\infty} f(y)\, dy = k \int_{-\infty}^{\infty} \frac{1}{1+y^2} dy$$

$$= k \cdot \arctan(y) \Big|_{-\infty}^{\infty} = k\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = k\pi$$



$$\Rightarrow k = \frac{1}{\pi}$$

So $\quad f(y) = \frac{1}{\pi} \frac{1}{1+y^2}.$

Note that

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{1+y^2}\, dy$$

$$= \frac{1}{\pi} \frac{1}{2} \ln(1+y^2) \Big|_{-\infty}^{\infty} = +\infty - \infty$$

← ~~Infinite~~ mean ~~mean it~~ does not exist

EXAMPLES

6.18 $Y \sim \text{Pareto}(\alpha, \beta)$, used to model incomes (Economics) has cdf:

for $\alpha > 0$, $\beta > 0$

$$F(y) = \begin{cases} 0 & , \quad y < \beta \\ 1 - (\beta/y)^{\alpha} & , \quad y \geq \beta \end{cases}$$

(a) Find a transform $G(U)$ so that $G(U) \sim Y$ when $U \sim \text{Unif}(0,1)$.

By the Prob Int. Transform, we know: $G(U) = F^{-1}(U)$.

$$u = F(y) = 1 - (\beta/y)^{\alpha} \quad \Rightarrow \quad \left(\frac{\beta}{y}\right)^{\alpha} = 1 - u$$

$$\Rightarrow \quad y = \beta(1-u)^{-1/\alpha}$$

So take: $G(U) = \beta(1-U)^{-1/\alpha}$.

(b) Show that $X = 1/Y$ has a power family dist with parameters α and $\theta = \beta^{-1}$, i.e.

$$F_X(x) = \begin{cases} 0 & , \quad x \leq 0 \\ (x/\theta)^{\alpha} & , \quad 0 \leq x \leq \theta \\ 1 & , \quad x > \theta. \end{cases}$$

Method of cdfs:

$$F_X(x) = P(X \leq x) = P\left(\frac{1}{Y} \leq x\right) = P\left(Y \geq \frac{1}{x}\right)$$

$$= 1 - F_Y(1/x)$$

$$= \begin{cases} \left(\frac{\beta}{1/x}\right)^{\alpha} & , \quad \frac{1}{x} \geq \beta \\ 1 & , \quad \frac{1}{x} < \beta \end{cases}$$

$$= \begin{cases} 0 & , \quad x \leq 0 \\ \left(\frac{x}{\theta}\right)^{\alpha} & , \quad 0 < x < \theta \\ 1 & , \quad x \geq \theta \end{cases}$$

, since when $x \leq 0$ would get -ve prob

, where $\theta = \frac{1}{\beta}$

check: pdf integrates to 1.

Ex: $Y \sim Beta(1, \theta)$. Show that $X = -\ln(1-Y) = h(Y)$ $\sim Exp(\beta = 1/\theta)$

Which method should we use? Clearly $supp\{X\} = (0, \infty)$

Have:

- $f(y) = \theta (1-y)^{\theta-1}, \quad 0<y<1$

  $\uparrow \dfrac{\Gamma(1+\theta)}{\Gamma(1)\Gamma(\theta)}$

- $F(y) = \int_0^y \theta(1-u)^{\theta-1} du = -(1-u)^\theta \Big|_0^y = 1-(1-y)^\theta, \quad 0<y<1$

- $m(t) = \ldots$ does not exist ...

1. Method of cdf's:

$$\begin{aligned}
F_X(x) &= P(X \le x) = P(-\ln(1-Y) \le x) \\
&= P(\ln(1-Y) \ge -x) \\
&= P(1-Y \ge e^{-x}) \\
&= P(Y \le 1-e^{-x}) \\
&= 1-[1-(1-e^{-x})]^\theta = 1-e^{-\theta x}, \quad x>0.
\end{aligned}$$

$\Rightarrow X \sim Exp(1/\theta)$.

2. Method of pdf's:

$x = -\ln(1-y) \Rightarrow e^{-x} = 1-y \Rightarrow y = 1-e^{-x}$.

$\Rightarrow \dfrac{dy}{dx} = e^{-x}$

$$\begin{aligned}
f_X(x) &= \left|\frac{dy}{dx}\right| f_Y(h^{-1}(x)) \\
&= |e^{-x}|\, \theta[1-(1-e^{-x})]^{\theta-1}, \quad 0<1-e^{-x}<1 \\
&= e^{-x}\, \theta\, e^{-x(\theta-1)}, \quad 0<e^{-x}<1 \\
&= \theta e^{-\theta x}, \quad x>0
\end{aligned}$$

$\Rightarrow X \sim Exp(1/\theta)$.

~~Even if m(t) existed would have hard time computing~~ $\cancel{m_X(t) = Ee^{tX} = Ee^{-t\ln(1-Y)}}$

3. Method of mgf's:

$$m_X(t) = \mathbb{E}\, e^{tX} = \mathbb{E}\, e^{-t\ln(1-Y)}$$

$$= \int_0^1 \theta(1-y)^{\theta-1-t}\, dy$$

$$= -\frac{\theta}{\theta-t}(1-y)^{\theta-t}\Big|_0^1 = \frac{\theta}{\theta-t}$$

$$= \frac{1}{1-\frac{1}{\theta}t}$$

$$\Rightarrow \quad X \sim \mathrm{Exp}(1/\theta).$$