Ch 5. Multivariate Probability Distributions

May have several random variables defined on the same sample space.

Ex. ① Select a person at random, measure height and weight (jointly continuous).

② Select a person at random, measure number of children and number of times voted in a presidential election (jointly discrete).

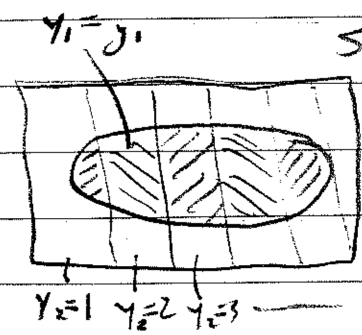
③ Smoke or not, lung cancer or not.

④ PH of water in lake, no. of species of fish

Def. If Y_1, Y_2 are discrete r.v.'s defined on the same sample space, then their joint probability function is

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2).$$

The marginal prob. fcn of Y_1 is



$$p_1(y_1) = P(Y_1 = y_1) = \sum_{y_2} p(y_1, y_2)$$

Similarly

$$p_2(y_2) = P(Y_2 = y_2) = \sum_{y_1} p(y_1, y_2)$$

Ex. A bridge hand consists of 13 cards selected at random (without replacement) from an ordinary 52 card deck.

Let $Y_1 = \#$ aces in the hand

$Y_2 = \#$ face cards in the hand.

There are 4 aces and 12 face cards in the deck.

So $0 \le Y_1 \le 4$; $0 \le Y_2 \le 12$; and of course $Y_1 + Y_2 \le 13$.

The jt pdf of Y_1 and Y_2 is

$$p(y_1, y_2) = \begin{cases} \dfrac{\binom{4}{y_1}\binom{12}{y_2}\binom{36}{13-y_1-y_2}}{\binom{52}{13}}, & \text{if } 0 \le y_1 \le 4,\ 0 \le y_2 \le 12, \text{ and } y_1 + y_2 \le 13. \\ 0, & \text{otherwise} \end{cases}$$

The marginal distributions are just hypergeometric:

$$p_1(y_1) = \frac{\binom{4}{y_1}\binom{48}{13-y_1}}{\binom{52}{13}}, \quad 0 \le y_1 \le 4$$

$$= \frac{\binom{4}{y_1}}{\binom{52}{13}} \sum_{y_2=0}^{13-y_1} \binom{12}{y_2}\binom{36}{13-y_1-y_2}$$

using:

$$\binom{m}{0}\binom{n}{p} + \cdots + \binom{m}{p}\binom{n}{0} = \binom{m+n}{p}$$

$$p_2(y_2) = \frac{\binom{12}{y_2}\binom{40}{13-y_2}}{\binom{52}{y_2}}, \quad 0 \le y_2 \le 12$$

Ex. Toss two fair dice. Y_1 = min of two up faces
(Die 1, Die 2) Y_2 = max ——————.

$p(y_1,y_2)$	y_2 (max) 1	2	3	4	5	6	$p_1(y_1)$
y_1 (min) 1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
2	0	1/36	2/36	2/36	2/36	2/36	9/36
3	0	0	1/36	2/36	2/36	2/36	7/36
4	0	0	0	1/36	2/36	2/36	5/36
5	0	0	0	0	1/36	2/36	3/36
6	0	0	0	0	0	1/36	1/36
$p_2(y_2)$	1/36	3/36	5/36	7/36	9/36	11/36	1

$p_1(y_1)$ hence the name "marginal dist^n".

$p_2(y_2)$

For $y_1, y_2 = 1, 2, \dots, 6$,

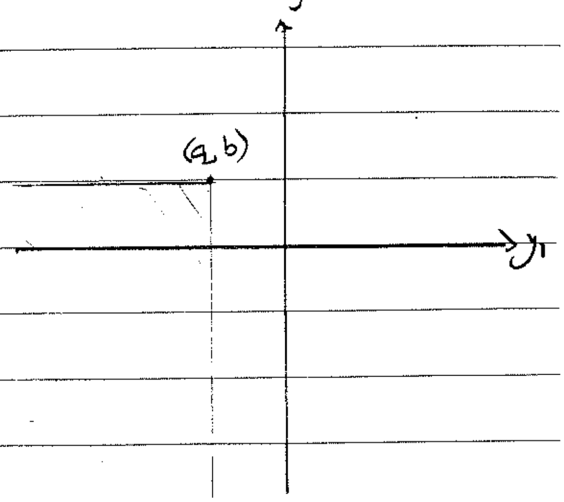
$$p(y_1,y_2) = \begin{cases} \frac{1}{36}, & \text{if } y_1 = y_2, \\ \frac{2}{36}, & \text{if } y_1 < y_2, \\ 0, & \text{o/w}. \end{cases}$$

Marginals:

$p_1(y_1)$ & $p_2(y_2)$ appear on margins.

Def. The joint (cumulative) distribution function of Y_1, Y_2 is

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2). \quad = \text{prob sw of } (y_1, y_2)$$



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Properties of the jt cdf

① $F(-\infty, -\infty) = F(-\infty, y_2) = F(y_1, -\infty) = 0.$

② $F(\infty, \infty) = 1.$

③ If $a \le b$ and $c \le d$, then

$P(a < Y_1 \le b,\ c < Y_2 \le d)$

$= F(b,d) - F(a,d) - F(b,c) + F(b,d).$

Note: For discrete r.v.'s,

$$F(a,b) = \sum_{y_1 \le a} \sum_{y_2 \le b} p(y_1, y_2)$$

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Ex. In the dice example ($Y_1 = \min$, $Y_2 = \max$),

$$F(2,4) = p(1,1) + p(1,2) + p(1,3) + p(1,4) + p(2,2) + p(2,3) + p(2,4) = \frac{12}{36} = \frac{1}{3}$$

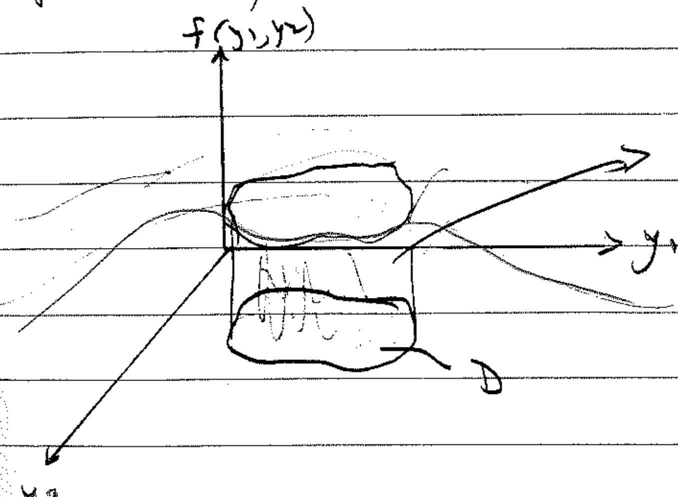
$$F(4,2) = \sim = \frac{4}{36} = \frac{1}{9}.$$

<u>Def</u>. Let Y_1, Y_2 be continuous r.v.'s with joint distribution function $F(y_1, y_2)$. If $\exists$ a nonnegative fcn $f(y_1,y_2)$ $\ni$

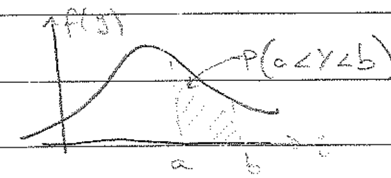
$$F(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(y_1,y_2)\,dy_1\,dy_2 \qquad \forall\ a,b,$$

then Y_1 and Y_2 are said to be <u>jointly continuous</u> and $f(y_1,y_2)$ is called the <u>joint prob. density fcn</u> of Y_1 and Y_2.

<u>Note</u>: If Y_1, Y_2 are jointly continuous, then the <u>volume</u> under their joint p.d.f. over a given region of the plane is the probability that (Y_1,Y_2) will lie in the region.



$\text{Volume} = P((Y_1,Y_2)\in D) = \iint_D f(y_1,y_2)\,dy_1\,dy_2$



3D analogue of this

<u>Properties of Joint P.d.f.</u>

① $f(y_1,y_2) \geq 0 \quad \forall\ y_1, y_2$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)\,dy_1\,dy_2 = 1.$

Ex. Suppose Y_1, Y_2 have joint p.d.f.

$$f(y_1, y_2) = k y_1 y_2, \quad 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1.$$

(a) Find k.

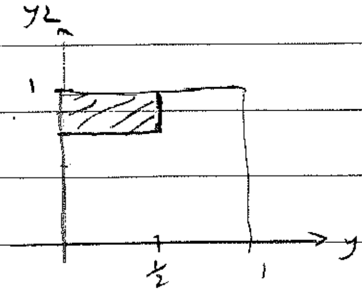
$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1\, dy_2 = \int_0^1\int_0^1 k y_1 y_2\, dy_1\, dy_2$$

$$= k \int_0^1 y_2 \left[\left(\tfrac{1}{2} y_1^2\right)\Big|_{y_1=0}^{1}\right] dy_2 = \tfrac{1}{2} k \int_0^1 y_2\, dy_2$$

$$= \tfrac{1}{2} k \left[\left(\tfrac{1}{2} y_2^2\right)\Big|_{y_2=0}^{1}\right] = \tfrac{1}{4} k$$

$$\Longrightarrow k = 4.$$

(b) Find $P(Y_1 \le \frac{1}{2}, Y_2 > \frac{3}{4})$



$$\int_0^{\frac{1}{2}}\int_{3/4}^{1} 4 y_1 y_2\, dy_2\, dy_1 = \int_0^{\frac{1}{2}} 2y_1 \left[\left(y_2^2\right)\Big|_{3/4}^{1}\right] dy_1$$

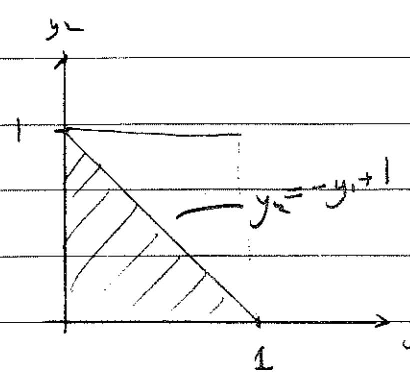
$$= \frac{7}{16} \int_0^{\frac{1}{2}} 2y_1\, dy_1 = \frac{7}{16}\left[\left(y_1^2\right)\Big|_0^{\frac{1}{2}}\right]$$

$$= \frac{7}{16} \cdot \frac{1}{4} = \frac{7}{64}$$

(c) Find $P(Y_1 + Y_2 \le 1)$.

$y_1 + y_2 \le 1$

$y_2 \le -y_1 + 1$



$$\int_0^1\int_0^{1-y_1} 4 y_1 y_2\, dy_2\, dy_1$$

$$= \int_0^1 2y_1 \left[y_2^2\Big|_{y_2=0}^{1-y_1}\right] dy_1 = \int_0^1 2y_1 (1-y_1)^2\, dy_1$$

$x = 1-y_1$, $dy_1 = -dx$

$$= \int_1^0 2(1-x)x^2(-dx) = \int_0^1 2x^2(1-x)\,dx = 2\int_0^1 (x^2 - x^3)\,dx = 2\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 2 \cdot \tfrac{1}{12} = \tfrac{1}{6}.$$

Do probs from §5.2.

§5.3 Marginal & Conditional Distributions

Def If Y_1 and Y_2 are jointly continuous with jt p.d.f. $f(y_1,y_2)$, then the marginal p.d.f. of Y_1 is

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1,y_2)\,dy_2 .$$

Similarly

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1,y_2)\,dy_1 .$$

Ex. In the last example,

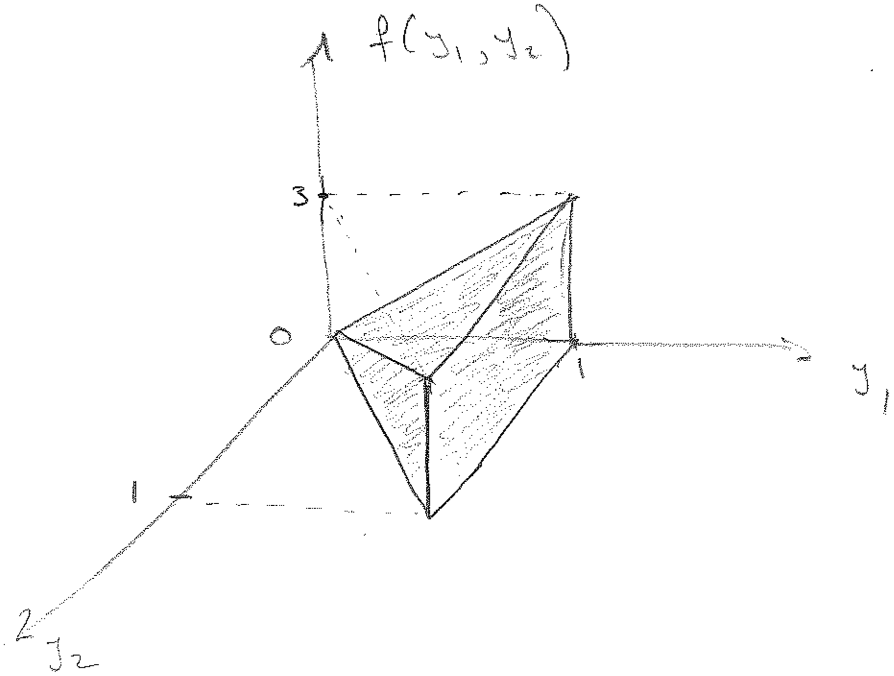
$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1,y_2)\,dy_2 = \int_0^1 4y_1y_2\,dy_2$$ doesn't involve y_2

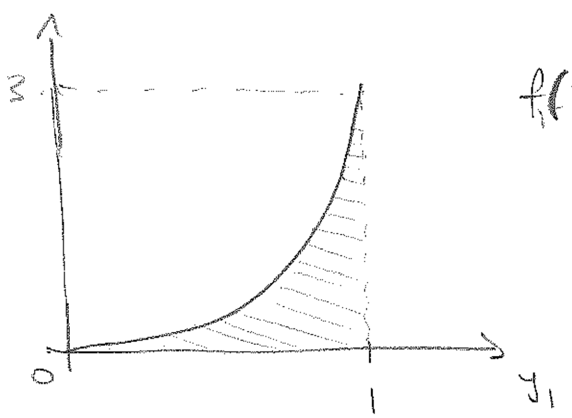
$$= 2y_1\left[\left(y_2^2\right)\Big|_0^1\right] = 2y_1, \quad 0 \le y_1 \le 1.$$

Similarly

$$f_2(y_2) = 2y_2, \quad 0 \le y_1 \le 1.$$

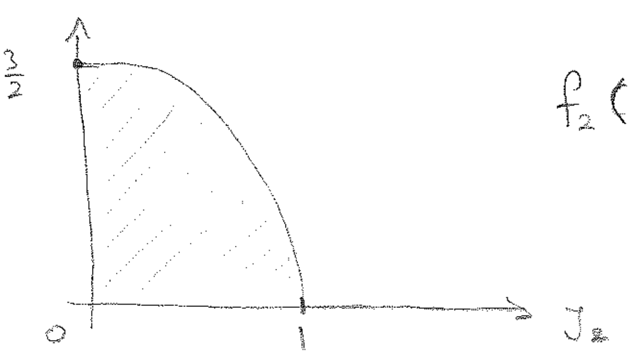
Maybe find $P(Y_1 \ge \frac{1}{2})$? Two ways?





$f_1(y_1) = 3y_1^2 \quad , \quad 0 \le y_1 \le 1$

Add up mass over y_2.

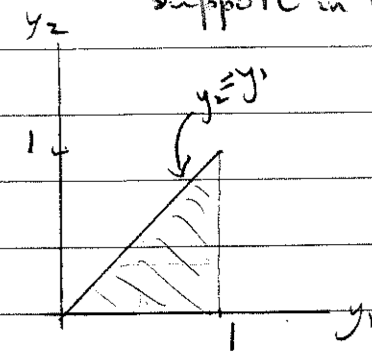


$f_2(y_2) = \frac{3}{2}\left(1 - y_2^2\right) \quad , \quad 0 \le y_2 \le 1$

Add up mass over y_1

Ex Let Y_1, Y_2 have jt pdf.

$$f(y_1, y_2) = 3y_1, \quad 0 \le y_2 \le y_1 \le 1.$$



(a) Verify that this is a density: $\int_0^1 \int_{y_2}^1 3y_1 \, dy_1 \, dy_2 = \frac{3}{2} \cdot \frac{2}{3} = 1.$

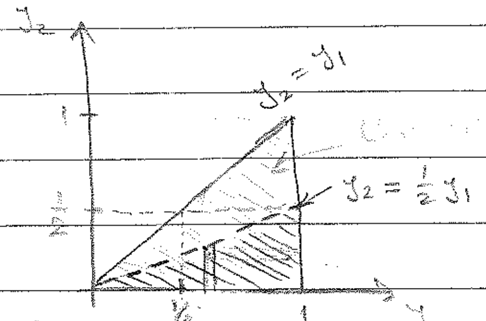
(b) Find $P(Y_1 < 2Y_2)$.

(c) Find the marginal pdf's of Y_1 and Y_2.

$f_1(y_1) = 3y_1^2, \quad 0 \le y_1 \le 1$ ← Does not involve y_2.

$f_2(y_2) = \frac{3}{2}(1 - y_2^2), \quad 0 \le y_2 \le 1.$ ← Does not involve y_1.

(b) $P(Y_1 < 2Y_2) = P(Y_2 > \frac{1}{2} Y_1)$



$$= \int_0^1 \int_{\frac{1}{2} y_1}^{y_1} 3y_1 \, dy_2 \, dy_1 = \frac{1}{2}$$

Change order of int.

$$= \int_0^{\frac{1}{2}} \int_{y_2}^{2y_2} 3y_1 \, dy_1 \, dy_2 + \int_{1/2}^1 \int_{y_2}^1 3y_1 \, dy_1 \, dy_2$$

(c) $f_1(y_1) = \int_0^{y_1} 3y_1 \, dy_2 = 3y_1^2, \quad 0 \le y_1 \le 1$

$f_2(y_2) = \int_{y_2}^1 3y_1 \, dy_1 = \frac{3}{2}(1 - y_2^2), \quad 0 \le y_2 \le 1$

<u>Def.</u> If Y_1 and Y_2 are jointly discrete r.v.'s, then the <u>conditional prob. fcn</u> of Y_1 given Y_2 is

$$p(y_1|y_2) = P(Y_1=y_1 | Y_2=y_2) = \frac{P(Y_1=y_1, Y_2=y_2)}{P(Y_2=y_2)} = \frac{p(y_1,y_2)}{p_2(y_2)}$$

provided that $p_2(y_2) > 0$.

Similarly,

$$p(y_2|y_1) = \frac{p(y_1,y_2)}{p_1(y_1)}, \quad \text{if } p_1(y_1) > 0.$$

<u>Def.</u> If Y_1 and Y_2 are jointly cts r.v.'s, then the <u>conditional density</u> of Y_1 given Y_2 is

Draw a picture.

$$f(y_1|y_2) = \begin{cases} f(y_1,y_2)/f_2(y_2), & \text{if } f_2(y_2) > 0, \\ 0, & \text{o/w.} \end{cases}$$

Similarly,

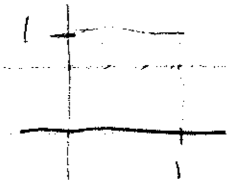
$$f(y_2|y_1) = \frac{f(y_1,y_2)}{f_1(y_1)} \quad \text{if } f_1(y_1) > 0.$$

<u>Note</u>: What do we mean by $p(3|2)$ or $f(3|2)$?
Notation is weak here. Probably should write

$$p_{Y_1|Y_2}(y_1|y_2) \quad \text{and} \quad f_{Y_1|Y_2}(y_1|y_2).$$

so $p_{Y_1|Y_2}(3|2)$ $\quad f_{Y_1|Y_2}(3|2)$.

$\Bigg\}$ ~~Book~~ Book uses $p(y_2=3 | y_1=5)$ etc.



Ex 1. $f(y_1,y_2) = \begin{cases} 4y_1y_2 & , \ 0 \le y_1 \le 1, \ 0 \le y_2 \le 1 \\ 0 & , \ o/w \end{cases}$

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1,y_2)\,dy_2 = \int_0^1 4y_1y_2\,dy_2 = 2y_1 \ , \ 0 \le y_1 \le 1$$

Similarly,

$$f_2(y_2) = \begin{cases} 2y_2 & , \ 0 \le y_1 \le 1 \\ 0 & , \ o/w \end{cases}$$

Also

$$f_1(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \begin{cases} \frac{4y_1y_2}{2y_2} = 2y_1 & , \ 0 \le y_1 \le 1, \ 0 \le y_2 \le 1 \\ 0 & , \ o/w \end{cases}$$

and

$$f(y_2|y_1) = \begin{cases} 2y_2 & , \ 0 \le y_1 \le 1, \ 0 \le y_2 \le 1 \\ 0 & , \ o/w \end{cases}$$

Note that for $0 \le y_2 \le 1$,

$$f(y_1|y_2) = f(y_1). \qquad \cancel{f(y_2|y_1) = f(y_2)}$$

Comment on independence.

So Y_1 is indep. of Y_2

So must also have: $f(y_2|y_1) = f(y_2)$ ✓

Ex 2. Recall

$$f(y_1, y_2) = 3y_1, \quad 0 \le y_2 \le y_1 \le 1$$

$$f_1(y_1) = 3y_1^2, \quad 0 \le y_1 \le 1$$

$$f_2(y_2) = \frac{3}{2}(1 - y_2^2), \quad 0 \le y_2 \le 1$$

Conditional densities are

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{3y_1}{\frac{3}{2}(1-y_2^2)} = \frac{2y_1}{1-y_2^2}, \quad 0 \le y_2 \le y_1 \le 1.$$

$$f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{3y_1}{3y_1^2} = \frac{1}{y_1}, \quad 0 \le y_2 \le y_1 \le 1.$$

Note that $\left.\begin{array}{l} \\ \text{" \quad "} \end{array}\right\}$ conditional on $Y_1 = y_1$, $Y_2 \sim U(0, y_1)$.

" " $Y_2 = y_2$, $Y_1 \sim \underset{\uparrow \atop \text{scaled}}{\text{Beta}}(\alpha = 2, \beta = 1)$

Ex 2. (ctd) Find

(i) $P(Y_1 \le \frac{1}{2} \mid Y_2 = \frac{1}{4}) =$

next page...

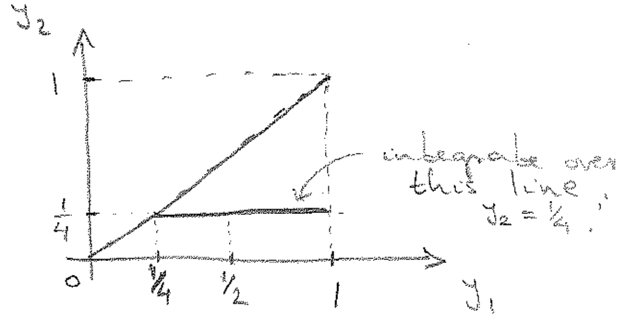
(ii) $P(Y_1 \le \frac{1}{2} \mid Y_2 \le \frac{1}{4}) =$

$P\left(Y_1 \le \frac{1}{2} \,\middle|\, Y_2 = \frac{1}{4}\right)$: $\quad f\left(y_1 \,\middle|\, y_2 = \frac{1}{4}\right) = \dfrac{3y_1}{\frac{3}{2}\left(1 - \frac{1}{16}\right)} = \dfrac{32}{15}\, y_1 ,$

$\frac{1}{4} \le y_1 \le 1 .$

$$= \int_{1/4}^{1/2} \frac{32}{15} y_1 \, dy_1$$

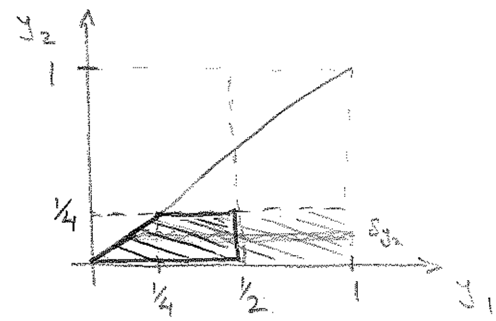
$$= \frac{1}{5}$$



and $\quad \int_{1/4}^{1} \frac{32}{15} y_1 \, dy_1 = 1 .$

$P\left(Y_1 \le \frac{1}{2} \,\middle|\, Y_2 \le \frac{1}{4}\right)$

$$= \frac{P\left(Y_1 \le \frac{1}{2},\, Y_2 \le \frac{1}{4}\right)}{P\left(Y_2 \le \frac{1}{4}\right)}$$



$$= \int_0^{1/4} \int_{y_2}^{1/2} 3y_1 \, dy_1 \, dy_2 \Bigg/ \underbrace{\int_0^{1/4} \int_{y_2}^{1} 3y_1 \, dy_1 \, dy_2}_{= \int_0^{\frac{1}{4}} \frac{3}{2}(1 - y_2^2)\, dy_2}$$

top

bottom

$$= \ldots$$

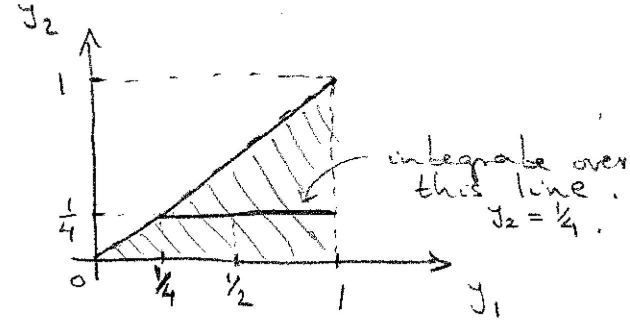
(i) $P\left(Y_1 \le \frac{1}{2} \mid Y_2 = \frac{1}{4}\right)$: $f\left(y_1 \mid y_2 = \frac{1}{4}\right) = \dfrac{3y_1}{\frac{3}{2}\left(1-\frac{1}{16}\right)} = \dfrac{32}{15} y_1$,

$\frac{1}{4} \le y_1 \le 1$.

$$= \int_{1/4}^{1/2} \frac{32}{15} y_1 \, dy_1$$

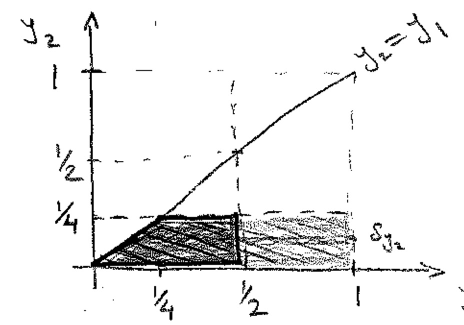
$$= \underline{\underline{\frac{1}{5}}} .$$

and $\displaystyle\int_{1/4}^{1} \frac{32}{15} y_1 \, dy_1 = 1$.



(ii) $P\left(Y_1 \le \frac{1}{2} \mid Y_2 \le \frac{1}{4}\right)$

$$= \frac{P\left(Y_1 \le \frac{1}{2},\ Y_2 \le \frac{1}{4}\right)}{P\left(Y_2 \le \frac{1}{4}\right)}$$



$$= \int_0^{1/4} \int_{y_2}^{1/2} 3y_1 \, dy_1 \, dy_2 \Bigg/ \underbrace{\int_0^{1/4} \int_{y_2}^{1} 3y_1 \, dy_1 \, dy_2}$$

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$$= \int_0^{\frac{1}{4}} \frac{3}{2}\left(1 - y_2^2\right) dy_2$$

$$= \int_0^{1/4} f_2(y_2) \, dy_2$$

$= \ldots$

Modeling with Conditional and Marginal Distributions

Marginal and Conditional distributions provide a very natural way to model the joint distribution of two r.v.s.

Note that

$$p(y_1, y_2) = p_1(y_1)\, p(y_2 | y_1) \qquad = p_2(y_2)\, p(y_1 | y_2)$$

and

$$f(y_1, y_2) = f_1(y_1)\, f(y_2 | y_1) \qquad = f_2(y_2)\, f(y_1 | y_2).$$

Ex. Let

Y_1 = # of traffic accidents on a highway in a given period

Y_2 = # of accidents involving fatalities for the same highway and period.

Possible model:

$Y_1 \sim Poi(\lambda = ?)$

$Y_2 | Y_1 = y_1 \sim Bin(n = y_1, p = ?)$

$$p(y_1, y_2) = p_1(y_1)\, p(y_2 | y_1)$$

$$= \left(\frac{\lambda^{y_1}}{y_1!} e^{-\lambda}\right) \binom{y_1}{y_2} p^{y_2} q^{y_1 - y_2}, \quad y_1 = 0, 1, 2, \ldots \quad y_2 = 0, 1, \ldots, y_1.$$

Do Probs 5.3

INDEPENDENCE

"Random variables Y_1 and Y_2, defined on the same sample space, are independent if any event involving Y_1 only is indep. of any event involving Y_2 only."

This is equivalent to ...

Y_1 and Y_2 are independent if $\forall$ $a<b$, $c<d$,

$$P(a \le Y_1 \le b, c \le Y_2 \le d) = P(a \le Y_1 \le b) \cdot P(c \le Y_2 \le d).$$

This is equivalent to

DEF. Y_1 & Y_2 are independent if

$$F(y_1, y_2) = F(y_1) \cdot F(y_2) \qquad \forall y_1, y_2.$$

Otherwise Y_1 and Y_2 are dependent.

Thm. Y_1 and Y_2 are independent iff

Discrete case: $p(y_1, y_2) = p_1(y_1) \cdot p_2(y_2) \quad \forall y_1, y_2$

Cts case: $f(y_1, y_2) = f_1(y_1) f_2(y_2) \quad \forall y_1, y_2.$

Note: If Y_1 and Y_2 are independent, then in the discrete case

$$p(y_1|y_2) := \frac{p(y_1, y_2)}{p_2(y_2)} = \frac{p_1(y_1) p_2(y_2)}{p_2(y_2)} = p_1(y_1) \quad \text{as long as } p_2(y_2) > 0$$

and

$$p(y_2|y_1) = p_2(y_2) \quad \text{as long as } p_1(y_1) > 0$$

and in the cts case

$$f(y_1|y_2) = f_1(y_1) \quad \text{as long as } f_2(y_2) > 0$$

and $f(y_2|y_1) = f_2(y_2)$ as long as $f_1(y_1) > 0$

y	$p_1(y)$	$p_2(y)$
1	11 →	1
2	9	3
3	7	5
4	5	7
5	3	9
6	1	11

$/36$

$Y_1 = \min(Y_1, Y_2)$

$Y_2 = \max(Y_1, Y_2)$ →

Can construct (Y_1, Y_2) with same marginals as above, but so that Y_1, Y_2 are indep:

Y_2

Y_1	1	2	3	4	5	6
1	11×1	11×3	11×5	11×7	11×9	11×11
2	9×1					
3	7×1					
4	5×1					
5	3×1					
6	1×1	–	–	–	–	1×11

$/36^2$

Ex. $Y_1 =$ min two dice, $Y_2 =$ max two dice. Dependent (give table): e.g. $p(1,1) = \frac{1}{36} \neq \frac{11}{36}\cdot\frac{1}{36} = p_1(1)\cdot p_2(1)$

Describe an experiment that yields two independent r.v.'s having the same marginal dist. as these. (marginals do not determine the joint dist).

Ex. Toss two fair dice, red and black. Let

$Y_1 =$ # on red die

$Y_2 =$ # on black die.

with our equally likely sample space,

$$p(y_1, y_2) = \frac{1}{36} \quad \forall\ y_1 = 1, \ldots, 6,\ y_2 = 1, \ldots, 6.$$

Also note that

$$p_1(y_1) = \frac{1}{6}, \quad y_1 = 1, \ldots, 6$$

and

$$p_2(y_2) = \frac{1}{6}, \quad y_2 = 1, \ldots, 6.$$

$\Rightarrow p(y_1, y_2) = p_1(y_1)\, p_2(y_2)$

So our equally likely model is equivalent to assuming each die is "fair," and that the outcomes are independent.

But $Y_1 =$ min, $Y_2 =$ max are dependent.

Ex.

$$f(y_1, y_2) = \begin{cases} 4y_1y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

We found earlier that

$$f_1(y_1) = \begin{cases} 2y_1, & 0 \le y_1 \le 1 \\ 0, & \text{o/w} \end{cases}$$

$$f_2(y_2) = \begin{cases} 2y_2, & 0 \le y_2 \le 1, \\ 0, & \text{o/w}. \end{cases}$$

Note that if $0 \le y_1 \le 1$ and $0 \le y_2 \le 1$, then $f(y_1, y_2) = 4y_1y_2 = f_1(y_1) f_2(y_2)$. Otherwise either $f_1(y_1) = 0$ or $f_2(y_2) = 0$ (or both) and $f_1(y_1) f_2(y_2) = 0 = f(y_1, y_2)$

So $f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)\ \forall\ y_1, y_2$, and Y_1 and Y_2 are independent.

Ex. $f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & o/w. \end{cases}$

Recall $f_1(y_1) = \begin{cases} 3y_1^2, & 0 \le y_1 \le 1, \\ 0, & o/w, \end{cases}$

$$f_2(y_2) = \begin{cases} \frac{3}{2}(1-y_2^2), & 0 \le y_2 \le 1, \\ 0, & o/w. \end{cases}$$

Since $f(y_1,y_2) \neq f_1(y_1) f_2(y_2)$, Y_1 and Y_2 are not independent.

Note: Marginals do ~~not~~ determine joint dist.

Ex. If $f(y_1, y_2) = \begin{cases} \frac{9}{2} y_1^2 (1-y_2^2), & 0 \le y_1 \le 1, 0 \le y_2 \le 1 \\ 0, & o/w \end{cases}$

then Y_1 and Y_2 are independent with the same marginals as the last example.

FACTORIZATION THRM

Thm Let Y_1 and Y_2 have joint density $f(y_1,y_2)$ which is positive on some rectangular region $[a \le y_1 \le b, c \le y_2 \le d]$ and 0 otherwise. Then Y_1 and Y_2 are independent iff

$$f(y_1,y_2) = g(y_1) \cdot h(y_2) \quad \forall\, a<y_1<b,\ c<y_2<d$$

where $g(y_1)$ is a nonnegative function of y_1 alone, and $h(y_2)$ is a nonnegative function of y_2 alone.

~~Mention Cartesian products!~~ In which case, $f_1(y_1) \propto g(y_1)$
$f_2(y_2) \propto h(y_2)$

Mention Indicator - functions

Ex. $f(y_1, y_2) = \begin{cases} 4y_1y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & o/w. \end{cases}$

$f(y_1, y_2)$ is positive on a rectangular region, and factors as fcns of y_1 and y_2 alone, e.g., $f(y_1,y_2) = \underbrace{(4y_1)}_{\sim \mathcal{B}(2,1)} \cdot \underbrace{(y_2)}_{\sim \mathcal{B}(2,1)}$, so Y_1 and Y_2 are independent.

Ex. $f(y_1, y_2) = \begin{cases} 2y_1 e^{-y_2}, & 0 \le y_1 \le 1;\ y_2 > 0, \\ 0, & o/w. \end{cases}$

Y_1 and Y_2 are independent. What are their marginal dists?

$\sim \mathcal{B}(2,1)$ $\sim Exp(1)$

Ex. $f(y_1, y_2) = \begin{cases} 2y_1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0, & o/w. \end{cases}$

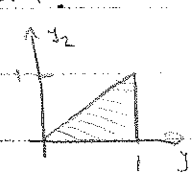
$f(y_1,y_2) = \underbrace{(2y_1)}_{\sim \mathcal{B}(2,1)} \cdot \underbrace{(1)}_{\sim U(0,1)} \implies Y_1$ and Y_2 are independent. What are their marginal dists?

Ex. $f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & o/w \end{cases}$

Y_1 and Y_2 are not independent.

But $f(y_1,y_2) = (3y_1)\cdot(1)$, so what's the problem?

The region where $f(y_1,y_2)$ is positive is not rectangular.



(Non-rectangular regions $\Rightarrow$ dependent)

§5.5 Expected Values

Discrete case: $E[g(Y_1,\dots,Y_k)] = \sum_{y_1}\cdots\sum_{y_k} g(y_1,\dots,y_k)\,p(y_1,\dots,y_k)$.

Continuous case: $E[g(Y_1,\dots,Y_k)] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(y_1,\dots,y_k)\,f(y_1,\dots,y_k)\,dy_1\cdots dy_k$.

Ex

$$f(y_1,y_2) = \begin{cases} 4y_1y_2, & 0\le y_1\le 1,\ 0\le y_2\le 1, \\ 0, & o/w. \end{cases}$$

Do geometric meth.?

$$E(Y_1Y_2) = \int_0^1\int_0^1 (y_1y_2)\cdot 4y_1y_2\,dy_1\,dy_2$$

$$= \int_0^1 4y_2^2\left(\int_0^1 y_1^2\,dy_1\right)dy_2$$

$$= \int_0^1 \tfrac{4}{3}y_2^2\,dy_2 = \frac{4}{9}$$

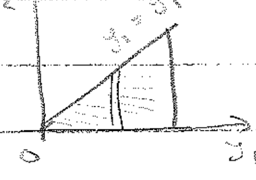
$$E\left(\frac{Y_2}{Y_1}\right) = \int_0^1\int_0^1 \frac{y_2}{y_1}\cdot 4y_1y_2\,dy_1\,dy_2$$

$$= \int_0^1\int_0^1 4y_2^2\,dy_1\,dy_2 = 4/3$$

Ex. $f(y_1,y_2) = \begin{cases} 3y_1, & 0\le y_2\le y_1\le 1, \\ 0, & \text{elsewhere}. \end{cases}$

$$E(Y_1Y_2) = \int_0^1\int_0^{y_1}(y_1y_2)(3y_1)\,dy_2\,dy_1 = \int_0^1 3y_1^2\left(\int_0^{y_1} y_2\,dy_2\right)dy_1$$

$$= \int_0^1 \tfrac{3}{2}y_1^4\,dy_1 = 3/10.$$



$$\frac{1}{2}$$

$$//$$

Ex. (ctd)

$$E\left(\frac{Y_2}{Y_1}\right) = \int_0^1 \int_0^{y_1} \left(\frac{y_2}{y_1}\right)(3y_1)\, dy_2\, dy_1$$

$$= \int_0^1 \int_0^{y_1} 3y_2\, dy_2\, dy_1 = \int_0^1 \tfrac{3}{2} y_1^2\, dy_1 = \tfrac{1}{2}.$$

~~Note:~~

$$\cancel{E[g(Y_1)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)\, f(y_1,y_2)\, dy_2\, dy_1}$$

$$\cancel{= \int_{-\infty}^{\infty} g(y_1)\left(\int_{-\infty}^{\infty} f(y_1,y_2)\, dy_2\right) dy_1}$$

$$\cancel{= \int_{-\infty}^{\infty} g(y_1)\, f_1(y_1)\, dy_1,}$$

~~so the new method extends (does not disagree with) the old univariate method of computing expectations.~~

Skip

THM. $E(c) = c.$

$$E[c\,g(Y_1,Y_2)] = c\,E[g(Y_1,Y_2)]$$

$$E[g_1(Y_1,Y_2) + \cdots + g_k(Y_1,Y_2)]$$
$$= E[g_1(Y_1,Y_2)] + \cdots + E[g_k(Y_1,Y_2)].$$

THM If Y_1 and Y_2 are independent, then

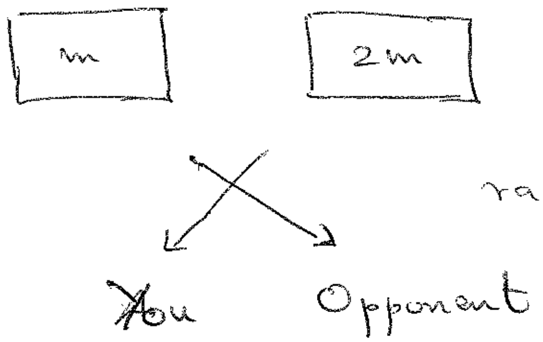
$$E[g(Y_1)\,h(Y_2)] = E[g(Y_1)] \cdot E[h(Y_2)],$$

provided the expectations exist.

$(E|\cdot| < \infty).$

The Exchange Paradox

- Two envelopes: \$m and \$2m (You don't know m).



~~You~~ randomly ~~assign~~ choose envelopes.

- You Open envelope, contains x dollars.
- Should you switch envelopes?
- You reason that if you switch you get:

  $\frac{x}{2}$ with prob $\frac{1}{2}$

  $2x$ " " $\frac{1}{2}$

- $\mathbb{E}(\text{switch}) = \frac{1}{2}\left(\frac{x}{2}\right) + \frac{1}{2}(2x) = \frac{5x}{4} > x$
- So you agree to a switch.
- But your opponent has done same calculation...
- ~~[illegible]~~ Christensen & Utts (1992), "Bayesian resolution of the exchange paradox", Amer. Statist., 46, 274-278.

Skip

Pf. (cts case)

$$E[g(Y_1)h(Y_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)h(y_2)f(y_1,y_2)\,dy_1\,dy_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)h(y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2 \quad \left(\text{by independence}\right)$$

$$= \int_{-\infty}^{\infty} h(y_2)f_2(y_2)\int_{-\infty}^{\infty} g(y_1)f_1(y_1)\,dy_1\,dy_2$$

$$= \left(\int_{-\infty}^{\infty} g(y_1)f_1(y_1)\,dy_1\right)\left(\int_{-\infty}^{\infty} h(y_2)f_2(y_2)\,dy_2\right)$$

$$= E[g(Y_1)]\cdot E[h(Y_2)].$$

Ex. For $f(y_1,y_2) = \begin{cases} 4y_1y_2, & 0\le y_1\le 1,\ 0\le y_2\le 1, \\ 0, & o/w. \end{cases}$

Recall $E(Y_1Y_2) = 4/9$.

Also, $f_1(y_1) = 2y_1, \quad 0\le y_1\le 1$

$f_2(y_2) = 2y_2, \quad 0\le y_2 \le 1,$

so

$$E(Y_1) = \int_0^1 y_1\cdot 2y_1\,dy_1 = 2/3$$

$$E(Y_2) = 2/3,$$

and of course,

$$E(Y_1Y_2) = E(Y_1)\cdot E(Y_2).$$

Ex: (Do this!)

Suppose $Y_1 \sim Beta(\alpha_1=2, \beta_1=1)$

$Y_2 \sim Exp(\beta_2=3)$

& $Y_1 \perp\!\!\!\perp Y_2$.

Find $E(Y_1Y_2)$.

$E(Y_1) = \frac{\alpha_1}{\alpha_1+\beta_1} = 2/3$

$E(Y_2) = \beta_2 = 3$

$E(Y_1Y_2) = E(Y_1)E(Y_2) = \frac{2}{3}\cdot 3 = 2$

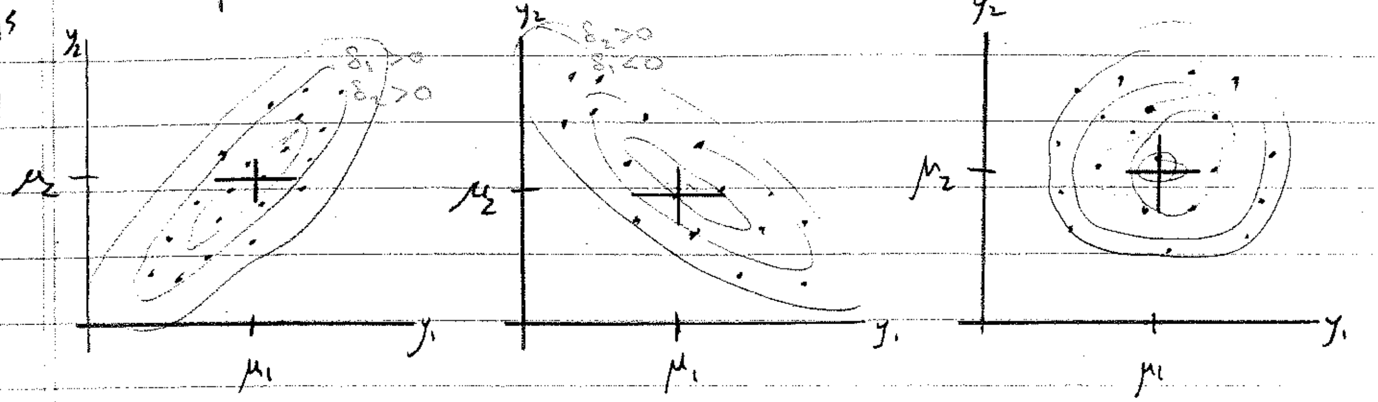
Do Probs 5.6

§5.7 COVARIANCE

Consider the function $(Y_1 - \mu_1)\cdot(Y_2-\mu_2)$ computed for each (Y_1, Y_2) in a random sample from the joint distribution.

Some possibilities are

Maybe draw contours of density first.



Positive association: Avg value of $(Y_1-\mu_1)(Y_2-\mu_2)$ is >0

Negative association: Avg value of $(Y_1-\mu_1)(Y_2-\mu_2)$ is <0

No association: Avg value of $(Y_1-\mu_1)(Y_2-\mu_2)$ is ≈ 0.

DEFN. The covariance of Y_1 and Y_2 is

$$Cov(Y_1, Y_2) = E\left[(Y_1-\mu_1)(Y_2-\mu_2)\right].$$

Covariance measures the linear dependence between Y_1 and Y_2.

Positive Cov indicates that large values of Y_1 tend to occur with large values of Y_2, and small values with small.

Neg. Cov indicates that large vals of one variable occur with small values of the other.

Note: $Cov(Y,Y) = V(Y)$.

Ex. Height and weight ~~have~~ have positive covariance.

weight and gas mileage of cars have neg. cov.

Note that $Cov(Y,Y) = V(Y)$.

Covariance is scale dependent: $Cov(aY_1+b, cY_2+d) = ac\, Cov(Y_1, Y_2)$.

(but location indep.)

DEFN. The correlation of Y_1 and Y_2 is

$$\rho_{Y_1,Y_2} = \frac{\mathrm{Cov}(Y_1,Y_2)}{\sigma_1\sigma_2} \quad \text{where } \sigma_1^2 = V(Y_1),\ \sigma_2^2 = V(Y_2).$$

NOTE. ρ is scale-free in the sense that if $X_1 = aY_1 + b$, $X_2 = cY_2 + d$, then

$$|\rho_{X_1,X_2}| = |\rho_{Y_1,Y_2}| \qquad \text{(Prove this.)} \qquad a>0,\ c>0$$

PROPERTIES OF ρ

(i) $-1 \le \rho \le 1$

(ii) $\rho = 1 \Rightarrow Y_2 = aY_1 + b, \quad a > 0.$

$\rho = -1 \Rightarrow Y_2 = aY_1 + b, \quad a < 0.$

(iii) $\rho > 0 \Rightarrow$ positive linear association.

$\rho < 0 \Rightarrow$ negative linear association.

Note: $\rho_{Y,Y} = 1$. Larger abs value of $\rho \Rightarrow$ stronger correlation.

THM. $\mathrm{Cov}(Y_1,Y_2) = E(Y_1Y_2) - \mu_1\mu_2$

Pf.
$$\begin{aligned}
\mathrm{Cov}(Y_1,Y_2) &= E[(Y_1-\mu_1)(Y_2-\mu_2)] \\
&= E[Y_1Y_2 - \mu_1Y_2 - \mu_2Y_1 + \mu_1\mu_2] \\
&= E(Y_1Y_2) - \mu_1E(Y_2) - \mu_2E(Y_1) + \mu_1\mu_2 \\
&= E(Y_1Y_2) - \mu_1\mu_2 - \mu_1\mu_2 + \mu_1\mu_2 \\
&= E(Y_1Y_2) - \mu_1\mu_2. \qquad \square
\end{aligned}$$

NOTE. Like $V(Y) = E[(Y-\mu)^2] = E(Y^2) - \mu^2$. In fact, $V(Y) = \mathrm{Cov}(Y,Y)$.

COR. If Y_1 and Y_2 are independent, then $\mathrm{Cov}(Y_1,Y_2) = 0$.
(Converse false!)

Proof: $\mathrm{Cov}(Y_1,Y_2) = E(Y_1Y_2) - \mu_1\mu_2 = \mu_1\mu_2 - \mu_1\mu_2 = 0$

Ex. $f(y_1,y_2) = 4y_1y_2$, $0 \le y_1 \le 1$, $0 \le y_2 \le 1$.

Since Y_1 and Y_2 are independent, $Cov(Y_1, Y_2) = 0$.
Note that we found in fact that

$$E(Y_1Y_2) = \frac{4}{9}, \quad E(Y_1) = \frac{2}{3} = E(Y_2),$$

so

$$Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = 0.$$

(and, of course, $\rho_{Y_1,Y_2} = 0$).

Ex. $f(y_1,y_2) = 3y_1$, $0 \le y_2 \le y_1 \le 1$.

We found $E(Y_1Y_2) = 3/10$,
and

$$f_1(y_1) = 3y_1^2, \quad 0 \le y_1 \le 1,$$

$$f_2(y_2) = \frac{3}{2}(1-y_2^2), \quad 0 \le y_2 \le 1$$

so

$$E(Y_1) = \int_0^1 y_1 \cdot 3y_1^2 \, dy_1 = \int_0^1 3y_1^3 \, dy_1 = 3/4$$

$$E(Y_2) = \frac{3}{2}\int_0^1 y_2(1-y_2^2)dy_2 = \frac{3}{2}\int_0^1 (y_2 - y_2^3)dy_2$$

$$= 3/2 \left(\frac{1}{2}y_2^2 - \frac{1}{4}y_2^4\right)\Big|_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = 3/8$$

$$\Rightarrow Cov(Y_1, Y_2) = E(Y_1Y_2) - \mu_1\mu_2 = \frac{3}{10} - \left(\frac{3}{4}\right)\left(\frac{3}{8}\right) = \underline{\underline{\frac{3}{160}}}$$

Also, $E(Y_1^2) = \int_0^1 y_1^2 \, 3y_1^2 \, dy_1 = 3/5 \Rightarrow V(Y_1) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$

$$E(Y_2^2) = \frac{3}{2}\int_0^1 y_2^2(1-y_2^2)\,dy_2 = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{5} \Rightarrow V(Y_2) = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$$

Ex (ct'd)

$$\Rightarrow \rho = \frac{Cov(Y_1,Y_2)}{\sigma_1\sigma_2} = \frac{3/160}{\sqrt{\frac{3}{80}}\cdot\sqrt{\frac{19}{320}}} = \frac{\cancel{3/160}}{\frac{\sqrt{3}\cdot\sqrt{19}}{\cancel{160}}}$$

$$= \boxed{\sqrt{\frac{3}{19}}} = \underline{.3974}$$

NOTE: Although Y_1 and Y_2 independent implies $Cov(Y_1,Y_2)=0$, the reverse implication does not hold. Consider the following example:

$p(y_1,y_2)$	y_2: -1	0	1	$p_1(y_1)$
y_1: -1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$p_2(y_2)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

Note that

$$E(Y_1)=0,\quad E(Y_2)=0,\quad E(Y_1Y_2)=0$$

$$\Rightarrow Cov(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0,$$

but Y_1 and Y_2 are not independent, since $p(y_1,y_2) \neq p_1(y_1)p_2(y_2)$.

~~What would this table have to look like in order that $Y_1 \perp\!\!\!\perp Y_2$?~~

Do Probs 5.7

§5.8 Means, Variances, and Covariances of Linear Combinations

Thm. For random variables $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_m$ defined on the same sample space, and constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$:

(i) $E\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i E(Y_i)$ (linearity)

(ii) $V\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum\sum_{1 \le i < j \le n} a_i a_j \operatorname{Cov}(Y_i, Y_j)$

(iii) $\operatorname{Cov}\left(\sum_{i=1}^{n} a_i Y_i, \sum_{j=1}^{m} b_j X_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \operatorname{Cov}(Y_i, X_j)$.

Example. If Y_1 and Y_2 have jt. p.d.f.

$$f(y_1, y_2) = 3y_1, \quad 0 \le y_2 \le y_1 \le 1$$

then we showed that

$$E(Y_1) = 3/4 \qquad V(Y_1) = 3/80 \qquad \operatorname{Cov}(Y_1, Y_2) = \frac{3}{160}.$$
$$E(Y_2) = 3/8 \qquad V(Y_2) = \frac{19}{320}$$

Suppose $X = Y_1 - Y_2$. Then

$$E(X) = E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$$

$$V(X) = V(Y_1 - Y_2) = V(Y_1) + V(Y_2) + 2(1)(-1)\operatorname{Cov}(Y_1, Y_2)$$
$$= \frac{3}{80} + \frac{19}{320} - \frac{2 \times 3}{160} = \frac{12 + 19 - \cancel{8}\,12}{320} = \frac{19}{320}$$

Cor. If $Y_1, \ldots, Y_n$ are independent, then

$$V\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i^2 V(Y_i).$$

Ex (Exercise #5.70). Suppose Y_1 and Y_2 have joint density

$$f(y_1, y_2) = \frac{1}{8} y_1 e^{-(y_1+y_2)/2}, \quad y_1 > 0,\ y_2 > 0,$$

and that $C = 50 + 2Y_1 + 4Y_2$.
Find $E(C)$ and $V(C)$.

Since
$$f(y_1, y_2) = \underbrace{\frac{1}{4}\, y_1 e^{-y_1/2}}_{\text{Gamma}(\alpha=2,\ \beta=2)} \cdot \underbrace{\frac{1}{2} \cdot e^{-y_2/2}}_{\text{Exp}(\beta=2)}, \quad y_1 > 0,\ y_2 > 0$$

we see that Y_1 and Y_2 are <u>independent</u> (by factorization theorem)

$$\Rightarrow \text{Cov}(Y_1, Y_2) = 0,$$

with

$$Y_1 \sim \text{Gamma}(\alpha=2, \beta=2) \Rightarrow E(Y_1) = 4, \quad V(Y_1) = 8$$
$$Y_2 \sim \text{Exp}(\beta=2) \Rightarrow E(Y_2) = 2, \quad V(Y_2) = 4.$$

Thus

$$E(C) = E(50 + 2Y_1 + 4Y_2) = 50 + 2E(Y_1) + 4E(Y_2)$$
$$= 50 + (2)(4) + (4)(2) = 66$$

and

$$V(C) = V(50 + 2Y_1 + 4Y_2) = V(2Y_1 + 4Y_2)$$
$$= 2^2 V(Y_1) + 4^2 V(Y_2) + 2(2)(4)\, \cancel{\text{Cov}(Y_1, Y_2)}^{\,0}$$
$$= (4)(8) + (16)(4) = 96.$$

Ex. $E(Y_1)=2$ $\quad E(Y_2)=-1$ $\quad E(Y_3)=4$

$V(Y_1)=4$ $\quad V(Y_2)=6$ $\quad V(Y_3)=8$

$\text{Cov}(Y_1,Y_2)=1$ $\quad \text{Cov}(Y_1,Y_3)=-1$ $\quad \text{Cov}(Y_2,Y_3)=0.$

$$U = 3Y_1+4Y_2-Y_3 \quad , \quad V = Y_1 - 2Y_2 .$$

$$E(U) = \cdots = -2$$

$$V(U) = \cdots = 170 = 9V(Y_1)+16V(Y_2)+V(Y_3)+2\left[12C(Y_1,Y_2)-3C(Y_1,Y_3)\right]$$

$$\text{Cov}(U,V) = \cdots = -37 = 3V(Y_1)-6C(Y_1,Y_2)+4C(Y_1,Y_2)-8V(Y_2)$$
$$-C(Y_1,Y_3)+2C(Y_2,Y_3)$$

Ex. Suppose $Y_1,\ldots,Y_n$ independent with

$$E(Y_i)=\mu \text{ and } V(Y_i)=\sigma^2 \quad \forall i=1,\ldots,n$$

all have same mean $\qquad$ all have same variance

and let

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i .$$

Then

$$E(\bar{Y}) = E\left(\frac{1}{n}\sum_{i=1}^n Y_i\right) = \frac{1}{n}\sum_{i=1}^n E(Y_i) = \frac{1}{n}\sum_{i=1}^n \mu = \frac{1}{n}\cdot n\mu = \mu,$$

and

$$V(\bar{Y}) = V\left(\frac{1}{n}\sum_{i=1}^n Y_i\right) = \frac{1}{n^2}V\left(\sum_{i=1}^n Y_i\right) = \frac{1}{n^2}\sum_{i=1}^n V(Y_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^n \sigma^2 = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n} .$$

Discuss statistical ramifications.

$\bar{Y}$ is unbiased for μ.

$V(\bar{Y}) \to 0$ as $n\to\infty$.

Do Probs 5.8

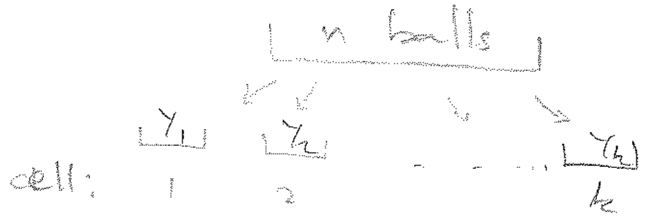
§5.9 The Multinomial Prob. Dist.

The Multinomial r.v. generalizes the Binomial to more than two outcomes (S, F).

Def (Multinomial Dist.)

1. n identical ~~trials~~ and indep. trials.
2. Outcome of each trial falls into one of k classes/cells.
3. p_i = prob that any one trial falls into cell i, $i = 1, \dots, k$.

   Also, $p_1 + p_2 + \cdots + p_k = 1$.
4. R.v.'s of interest are $Y_1, \dots, Y_k$, where
   Y_i = # of trials for which the outcome falls into cell i.
   Also, $Y_1 + \cdots + Y_k = n$.



Joint prob. fcn of $Y_1, \dots, Y_k$ is:

$$p(y_1, \dots, y_k) = \frac{n!}{y_1! \cdots y_k!} p_1^{y_1} \cdots p_k^{y_k}$$

where $\sum_1^k p_i = 1$, $\sum_1^k y_i = n$.

Theom: If $(Y_1, \dots, Y_k) \sim$ Multinomial,

(i) $EY_i = np_i$

(ii) $V(Y_i) = np_i(1-p_i)$

(iii) $\text{Cov}(Y_i, Y_j) = -np_ip_j$, $i \neq j$.

Note: If $k=2$, Y_1 is a Binomial r.v.

Ex: From Census figures:

Age	prop of adult pop	
18–24	.18	1
25–34	.23	2
35–44	.16	0
45–64	.27	2
≥ 65	.16	0

If 5 adults randomly sampled, prob

$$p(1,2,0,2,0) = \frac{5!}{1!\,2!\,0!\,2!\,0!}(.18)^1(.23)^2(.16)^0(.27)^2(.16)^0$$

$$= .0208$$

Do Probs 5.9

§ 5.11 CONDITIONAL EXPECTATIONS

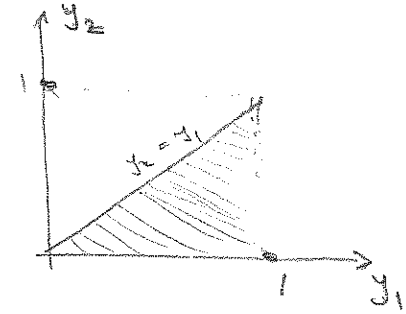
Recall : Y_1, Y_2 jointly cont., $f(y_1, y_2)$ with cond. dist: $f(y_1 | Y_2 = y_2)$.

Then : $$\mathbb{E}[g(Y_1) | Y_2 = y_2] = \int_{-\infty}^{\infty} g(y_1)\, f(y_1 | Y_2 = y_2)\, dy_1$$

If Y_1, Y_2 jointly discrete: $p(y_1, y_2)$, $p(y_1 | Y_2 = y_2)$

$$\mathbb{E}[g(Y_1) | Y_2 = y_2] = \sum_{y_1} g(y_1)\, p(y_1 | y_2)$$

Ex : $$f(y_1, y_2) = \begin{cases} 3y_1 , & 0 \le y_2 \le y_1 \le 1 \\ 0 , & o/w \end{cases}$$



Recall : $$f(y_1 | y_2) = \frac{2y_1}{1 - y_2^2} , \quad 0 \le y_2 \le y_1 \le 1$$

$$\mathbb{E}[Y_1 | Y_2 = y_2] = \int_{-\infty}^{\infty} y_1\, f(y_1 | y_2)\, dy_1$$

$$= \int_{y_2}^{1} \frac{2y_1^2}{1 - y_2^2}\, dy_1 = \frac{2(1 - y_2^3)}{3(1 - y_2^2)} , \quad 0 \le y_2 \le 1$$

Notice how $\mathbb{E}[Y_1 | Y_2 = y_2]$ is a function of y_2 ($\because y_1$ has been integrated out). Letting Y_2 range over all its values, we can think of $\mathbb{E}(Y_1 | Y_2)$ as a function of the r.v. Y_2. E.g. in above example:

$$\mathbb{E}(Y_1 | Y_2) = \frac{2(1 - Y_2^3)}{3(1 - Y_2^2)}$$

$\Rightarrow$ it has a mean & variance & is itself a r.v. !

Theom (Conditional mean formula)

$$E[Y_1] = \underset{\substack{\uparrow \\ \text{expect. w.r.t. } f_2(y_2)}}{E}\Big[\underset{\substack{\uparrow \\ \text{expect. w.r.t. } f(y_1|y_2)}}{E}(Y_1|Y_2)\Big]$$

Theom (Conditional variance formula)

$$\cancel{V\big[E(Y_1|Y_2)\big] = V\big[Y_1\big] - E\big[V(Y_1|Y_2)\big]}$$

or

$$V[Y_1] = V\big[E(Y_1|Y_2)\big] + E\big[V(Y_1|Y_2)\big].$$

Ex: Suppose we know:

$$Y_1|Y_2 \sim N(Y_2\,,\,3Y_2^2) \quad \text{and} \quad Y_2 \sim \text{Gamma}(\alpha, \beta)$$

What is $E(Y_1)$ and $V(Y_1)$?

$$E[Y_1] = E\big[E(Y_1|Y_2)\big] = E[Y_2] = \alpha\beta.$$

$$\begin{aligned} V[Y_1] &= V\big[E(Y_1|Y_2)\big] + E\big[V(Y_1|Y_2)\big] \\ &= V[Y_2] + E[3Y_2^2] \\ &= \alpha\beta^2 + 3E[Y_2^2] \qquad , \qquad EY_2^2 = V(Y_2) + (EY_2)^2 = \alpha\beta^2 + \alpha^2\beta^2 = \alpha\beta^2(1+\alpha) \\ &= \underline{\underline{\alpha\beta^2 + 3\alpha\beta^2(1+\alpha)}}. \end{aligned}$$

easier than finding marginal for Y_1 & computing $E(Y_1)$ & $V(Y_1)$ via integration.

Do Probs 5-11