CHAPTER 4

DISTRIBUTION FUNCTIONS AND CONTINUOUS DISTRIBUTIONS

(c.d.f)

DEFN. For any r.v. Y, the (cumulative) distribution function of Y, denoted $F(y)$, is given by

$$F(y) = P(Y \le y), \qquad -\infty < y < \infty.$$

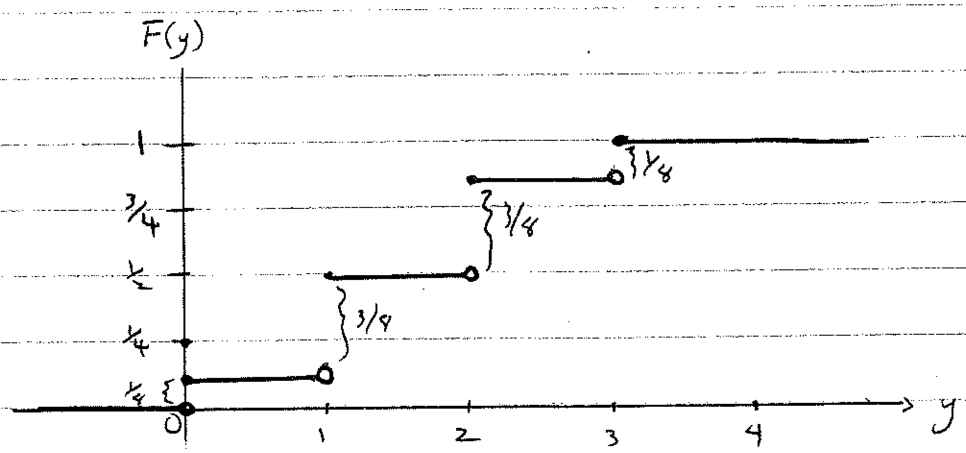
Ex. Let $Y \sim Bin(3, \frac{1}{2})$.

Then

y	$p(y)$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$

and

$$F(y) = P(Y \le y) = \begin{cases} 0, & \text{if } y < 0, \\ \frac{1}{8}, & \text{if } 0 \le y < 1, \\ 4/8, & \text{if } 1 \le y < 2, \\ 7/8, & \text{if } 2 \le y < 3, \\ 1, & \text{if } 3 \le y. \end{cases}$$



For discrete Y,

$$F(y) = \sum_{y' \le y} p(y').$$

PROPERTIES 1. $\lim_{y \to -\infty} F(y) =: F(-\infty) = 0$

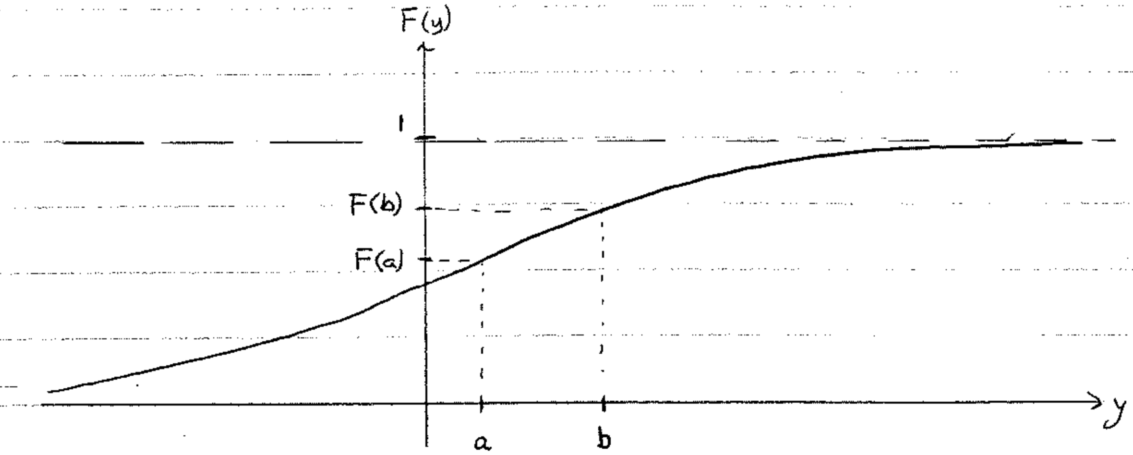
2. $\lim_{y \to \infty} F(y) =: F(\infty) = 1$

3. $F(a) \le F(b)$ if $a \le b$ (Non decreasing)

If $a \le b$, then

NOTE: $P(a < Y \le b) = F(b) - F(a)$. Also $P(Y=y) = \Delta F(y) = F(y) - \lim_{y' \uparrow y} F(y')$.

DEFN. If the distribution function of a r.v. Y is continuous, then Y is said to be a continuous r.v.



$$P(a < Y \le b) = P(Y \le b) - P(Y \le a) = F(b) - F(a), \quad a \le b.$$

NOTE. For a cts r.v. Y, $P(Y=y) = 0 \quad \forall y, \ -\infty < y < \infty$.

Why?

Because if $P(Y=y_0) > 0$, then $F(y)$ has a jump at $y = y_0$, and is not cts. $P(Y=y_0) = P(Y \le y_0) - P(Y < y_0) = F(y_0) - \lim_{y \uparrow y_0} F(y)$

Omit

~~DEFN. A r.v. Y is said to be absolutely continuous if F(y) is continuous and F(y) is differentiable at all but at most countably many points.~~ almost everywhere

The derivative of F,

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

is called the probability density function (p.d.f.) of Y. (wherever the derivative exists)

For us, all continuous distributions will have piecewise differentiable cdf's. The derivative is the density of the dist.

~~Note that~~

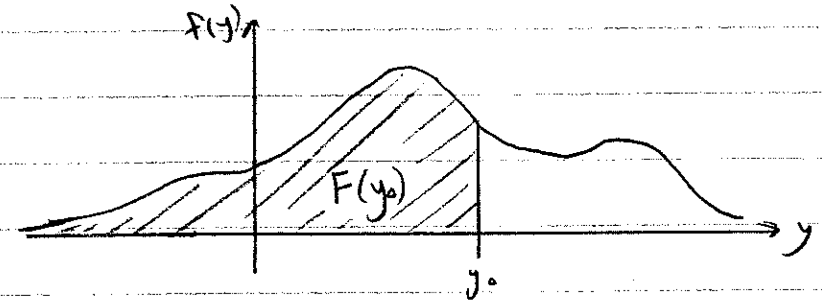
$$\cancel{F(b)-F(a) = \int_a^b f(y)\,dy.}$$

~~In particular,~~

$$\cancel{F(y) = \int_{-\infty}^{y} f(u)\,du.}$$

The p.d.f. is just a theoretical prob model for a population of measurements. It is the natural abstraction of a histogram. For this reason we are more likely to describe cts distns in terms of their p.d.f. than in terms of the c.d.f.

The p.d.f. is the analogue of the p.f. for continuous discrete r.v.'s.



~~Again~~, Note that:

$$F(b) - F(a) = \int_a^b f(u)\,du = P(a < Y < b)$$

Letting $a \to -\infty$ gives

$$F(b) = \int_{-\infty}^{b} f(u)\,du$$

and letting $b \to +\infty$ gives

$$\int_{-\infty}^{\infty} f(u)\,du = 1.$$

Also, since F is nondecreasing,

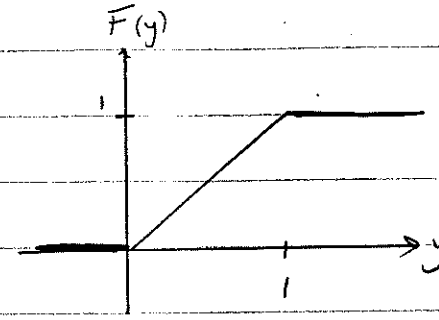
$$f(y) \geq 0 \quad \forall y.$$

(*) Properties of the p.d.f.: ① $f(y) \geq 0 \quad \forall y$

② $\int_{-\infty}^{\infty} f(y)\,dy = 1$.

Ex. $\mathfrak{f}$

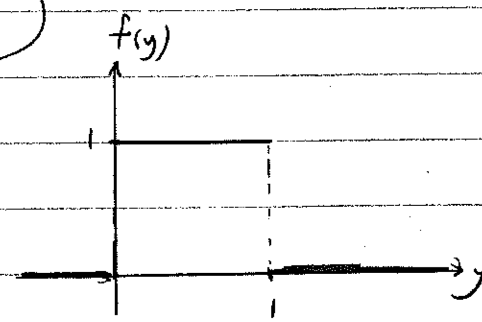
$$F(y) = \begin{cases} 0, & y < 0, \\ y, & 0 \leq y < 1, \\ 1, & y \geq 1. \end{cases}$$



The p.d.f. of F is

$$f(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{o/w} \end{cases}$$

Support of Y



or just

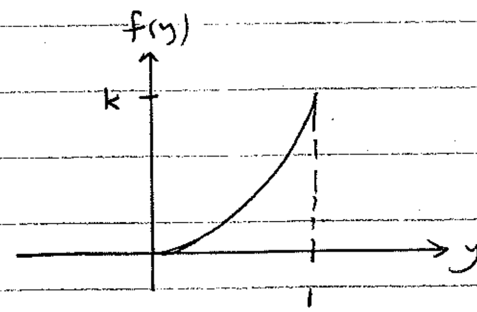
$f(y) = 1, \quad 0 \leq y \leq 1$.

Called the uniform distribution on (0,1), because the probabability is spread uniformly over the interval (0,1).

Ex. Let Y be a cts r.v. with p.d.f.

Support of Y.

$f(y) = ky^2, \quad 0 \leq y \leq 1$



where k is a constant.

① Since f is a p.d.f.,

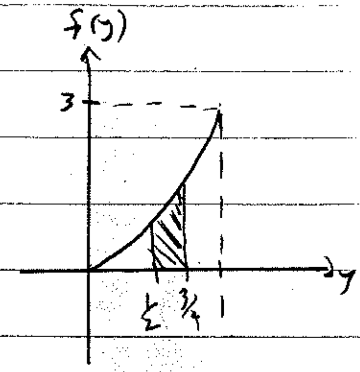
$$1 = \int_{-\infty}^{\infty} f(y)\,dy = \int_0^1 ky^2\,dy = \tfrac{1}{3}ky^3\Big|_0^1 = \tfrac{1}{3}k$$

$\Rightarrow k = 3 \Rightarrow f(y) = 3y^2, \quad 0 \leq y \leq 1$.

(2) Find $P(\frac{1}{2} < Y < \frac{3}{4})$.

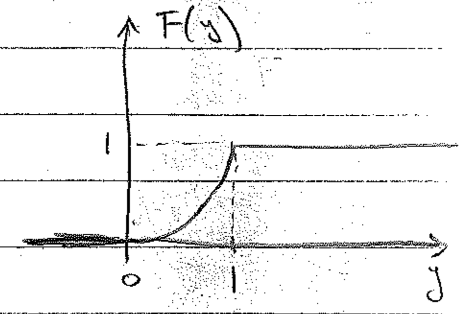
$$P\left(\tfrac{1}{2} < Y < \tfrac{3}{4}\right) = \int_{\frac{1}{2}}^{3/4} f(y)\,dy = \int_{\frac{1}{2}}^{3/4} 3y^2\,dy$$

$$= y^3\Big|_{\frac{1}{2}}^{3/4} = \frac{27}{64} - \frac{1}{8} = \boxed{\frac{19}{64}}$$



(3) Find the c.d.f., $F(y)$:

$$F(y) = \begin{cases} 0 & , \quad y \le 0 \\ \int_0^y 3y^2\,dy = y^3 & , \quad 0 \le y \le 1 \\ 1 & , \quad y \ge 1 \end{cases}$$



Note:

$$P\left(\tfrac{1}{2} < Y < \tfrac{3}{4}\right) = F\left(\tfrac{3}{4}\right) - F\left(\tfrac{1}{2}\right) = \frac{27}{64} - \frac{1}{8} = \frac{19}{64}.$$

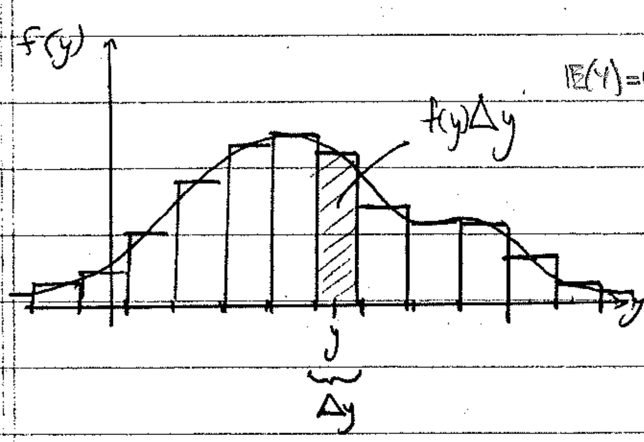
Add Exercise 4.13 to HMWK!!!

EXPECTATIONS.

The exp. of a cts r.v. Y with density $f(y)$ is defined to be

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy.$$

NOTE: This is just the continuous analogue of $E(Y) = \sum_y y p(y)$.



$\mathbb{E}(Y)$ = Center of mass

$$= \sum_y y f(y)\, \Delta y \longrightarrow \int_{-\infty}^{\infty} y f(y)\, dy$$

as $\max|\Delta y| \to 0$.

Ex. In the case of the $U(0,1)$ dist,

$$f(y) = 1, \quad 0 \le y \le 1,$$

and

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy = \int_0^1 y\, dy = \boxed{\tfrac{1}{2}}$$

Ex. In the previous example,

$$f(y) = 3y^2, \quad 0 \le y \le 1,$$

$$E(Y) = \int_0^1 y(3y^2)\, dy = \int_0^1 3y^3\, dy = \tfrac{3}{4} y^4 \Big|_0^1 = \boxed{3/4}$$

NOTE: Compare the graphs of the two f's in these last two examples.

THM. (LAW OF THE UNCONCIOUS STATISTICIAN – CTS VERSION)

If $g(y)$ is a real-valued fcn, then

$$E(g(Y)) = \int_{-\infty}^{\infty} g(y) \cdot f(y)\,dy.$$

THM. (LINEARITY OF EXPECTATION)

(i) $E(c) = c$

(ii) $E[c\,g(Y)] = c \cdot E[g(Y)]$

(iii) $E[g_1(Y) + \cdots + g_k(Y)] = E[g_1(Y)] + \cdots + E[g_k(Y)]$

NOTE. we still define

$$V(Y) = E[(Y-\mu)^2]$$

and we still have

THM: $V(Y) = E(Y^2) - [E(Y)]^2 = E(Y^2) - \mu^2$,

THM: If $\mu = E(Y)$ & $\sigma^2 = V(Y)$, then $E(aY+b) = a\mu + b$ and $V(Y) = a^2\sigma^2$.

proofs are identical.

EX. For the $U(0,1)$ dist, $f(y) = 1$, $0 \le y \le 1$, and

$$E(Y^2) = \int_0^1 y^2\,dy = \frac{1}{3}$$

and

$$V(Y) = E(Y^2) - \mu^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} = .0833$$

$E[2(y+1)(y-2)]$.

EX. For $f(y) = 3y^2$, $0 \le y \le 1$,

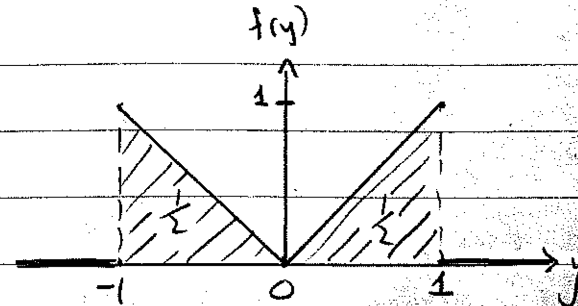
$$E(Y^2) = \int_0^1 y^2 \cdot 3y^2\,dy = \int_0^1 3y^4\,dy = 3/5$$

$$\Rightarrow V(Y) = 3/5 - (3/4)^2 = 3/5 - \frac{9}{16} = \frac{48-45}{80} = \boxed{\frac{3}{80}} = .0375$$

Again, compare.

Ex. Let Y have p.d.f. $f(y) = |y|$, $-1 \le y \le 1$.

Note that $f(y) \ge 0 \ \forall y$, and



$$\int_{-\infty}^{\infty} f(y)\,dy = \int_{-1}^{0} (-y)\,dy + \int_{0}^{1} y\,dy$$

$$= \tfrac{1}{2} + \tfrac{1}{2} = 1$$

Here,

$$\mu = E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_{-1}^{0} y\cdot(-y)\,dy + \int_{0}^{1} y\cdot y\,dy$$

$$= \int_{-1}^{0} (-y^2)\,dy + \int_{0}^{1} y^2\,dy = -\tfrac{1}{3}y^3\Big|_{-1}^{0} + \tfrac{1}{3}y^3\Big|_{0}^{1}$$

$$= -\tfrac{1}{3} + \tfrac{1}{3} = \boxed{0}$$ ← Of course. Mention symmetric densities and their means.

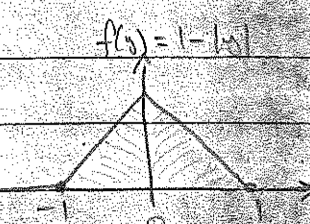
$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_{-1}^{0} y^2\cdot(-y)\,dy + \int_{0}^{1} y^2\cdot y\,dy$$

$$= \int_{-1}^{0} (-y^3)\,dy + \int_{0}^{1} y^3\,dy$$

$$= -\tfrac{1}{4}y^4\Big|_{-1}^{0} + \tfrac{1}{4}y^4\Big|_{0}^{1} = \tfrac{1}{4} + \tfrac{1}{4} = \boxed{\tfrac{1}{2}}$$

$$\sigma^2 = V(Y) = E(Y^2) - \mu^2 = \tfrac{1}{2} - 0^2 = \boxed{\tfrac{1}{2}}$$

Ask them to work this out for



$$f(y) = 1 - |y|, \quad -1 \le y \le 1.$$

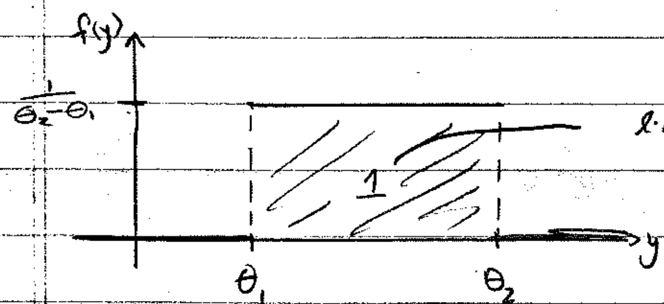
What do they expect? $EY = 0$, $V(Y) = 1/6$

§4.4 The Uniform Probability Distribution

Defn. Let $\theta_1 < \theta_2$ be real no.'s. The uniform dist'n on the interval (θ_1, θ_2) has p.d.f.

$$f(y) = \frac{1}{\theta_2 - \theta_1}, \qquad \theta_1 < y < \theta_2.$$

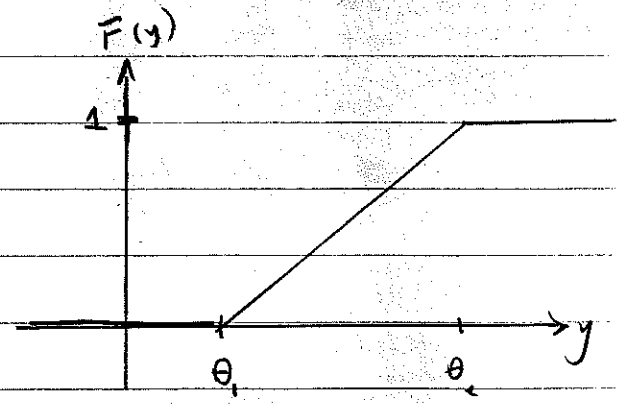
Note: This is a two parameter family of dist's (Parameters are θ_1 and θ_2).



$$l \cdot h = (\theta_2 - \theta_1) \cdot \frac{1}{\theta_2 - \theta_1} = 1.$$

Calculate the C.D.F.

$$F(y) = \begin{cases} 0, & y \le \theta_1, \\ \dfrac{y - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2, \\ 1, & y \ge \theta_2. \end{cases}$$



Note: THE uniform (0,1) dist. is the one with $\theta_1 = 0$, $\theta_2 = 1$.

Thm If $Y \sim U(\theta_1, \theta_2)$, then

$$\mu = E(Y) = \frac{\theta_1 + \theta_2}{2} \quad \text{and} \quad \sigma^2 = V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}.$$

Give intuition about this!

Proof. Exercise. (See text for μ).

$U \sim U(0,1)$
$Y = (\theta_2 - \theta_1)U + \theta_1$. Then use $E(Y) = aE(U) + b$; $V(Y) = a^2 V(U)$

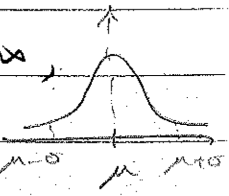
Ex ... ?

write this as $-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2$

4.5 The Normal Probability Dist

Defn. A r.v. Y is said to have a normal probability dist. if Y has p.d.f.

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(y-\mu)^2/2\sigma^2\right], \quad -\infty < y < \infty,$$



where $-\infty < \mu < \infty$ and $\sigma > 0$. (Two-parameter family).

Thm. If Y is a normally dist'd r.v. with parameters μ and σ, then

$$E(Y) = \mu \quad \text{and} \quad V(Y) = \sigma^2.$$

Proof. Later.

Exs: (1) Physical measurements, (2) scores on exams, (3) everywhere...

Typically write $Y \sim N(\mu, \sigma^2)$ to indicate that Y is normally dist'd with mean μ and variance σ^2. Draw density, show μ and pts of inflection.

Note: It is tricky to show that $\int_{-\infty}^{\infty} f(y)\,dy = 1$ for a normal density.

To simplify, try it for the standard normal distn, $N(0,1)$.

The p.d.f. is then

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy.$$

Now,

$$\int_{-\infty}^{\infty} f(y)\,dy = 1 \iff \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}.$$

$$\iff \left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)^2 = 2\pi.$$

But

$$\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)$$

$$= \int\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy \quad \text{[Now transform to polar coordinates.]}$$

$x = r\cos\theta$, $y = r\sin\theta$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta = 2\pi$$

Note: To evaluate probabilities for Y, we need to be able to compute integrals like

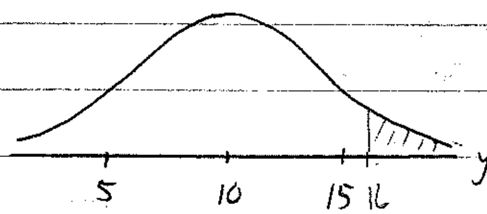
$$P(a \le Y \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2}\,dy$$

$z = \frac{y-\mu}{\sigma}$ change of variables reduces this to $\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$ but...

There is no closed form for this integral. Must be evaluated numerically or with tables. (Table 4)

Thm. If $Y \sim N(\mu, \sigma^2)$, and $Z = \frac{Y-\mu}{\sigma}$, then $Z \sim N(0,1)$ [standard normal].

Proof. (Later)

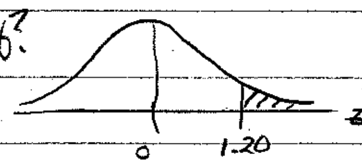


Ex. Suppose $Y \sim N(10, 25)$.

① Find $P(Y > 16)$.

$$P(Y>16) = P\left(\frac{Y-10}{5} > \frac{16-10}{5}\right) = P(Z > 1.20) = .1151$$

z-score of 16 is 1.20. What does it represent?

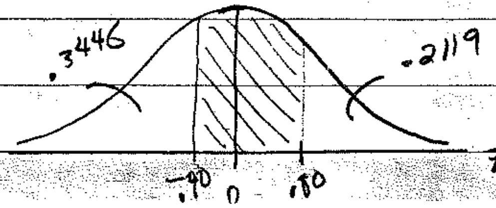


② Find $P(8 \le Y \le 14)$.

$$P(8 \le Y \le 14) = P\left(\frac{8-10}{5} \le \frac{Y-10}{5} \le \frac{14-10}{5}\right) = P(-.40 \le Z \le .80)$$

$= 1 - .3446 - .2119$

$= \underline{\underline{.4435}}$

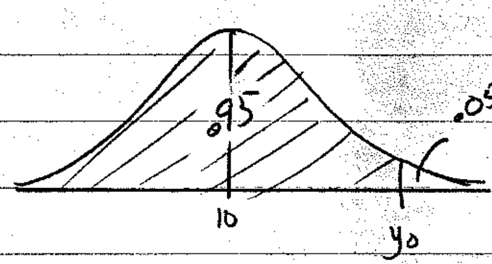


NOTE.

RECALL EMPIRICAL RULE

$$\begin{cases} P(\mu-\sigma \le Y \le \mu+\sigma) = P(-1 \le Z \le 1) = 1 - 2\times .1587 = .6826 \approx 68\% \\ P(\mu-2\sigma \le Y \le \mu+2\sigma) = P(-2 \le Z \le 2) = 1 - 2\times .0228 = .9544 \approx 95\% \\ P(\mu-3\sigma \le Y \le \mu+3\sigma) = P(-3 \le Z \le 3) = 1 - 2\times .00135 = .9973 \approx 100\% \end{cases}$$

Ex. In the last example, find the 95th percentile of the dist'n of Y.



$$.05 = P(Y > y_0)$$
$$= P\left(\frac{Y-10}{5} > \frac{y_0-10}{5}\right)$$
$$= P(Z > z_0)$$

where $z_0 = \dfrac{y_0 - 10}{5}$.

By table, $z_0 = 1.645 \Rightarrow \dfrac{y_0-10}{5} = 1.645$

$$\Rightarrow y_0 = 10 + 5\times 1.645 = \boxed{18.225}$$

Ex. Suppose a machine dispenses coffee with the mean amount determined by a control setting and standard deviation equal to .25 oz.
What should be the mean in order that only 2% of 5 oz cups are <u>under</u>filled?
Assume the amount dispensed into a given cup follows a normal dist.

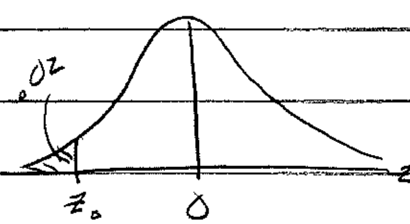
$Y \sim N(\mu, \sigma = .25)$, Y = oz dispensed by machine into any given cup.

want

$$.02 = P(Y < 5) = P\left(\frac{Y-\mu}{.25} < \frac{5-\mu}{.25}\right)$$

$$= P\left(Z < \frac{5-\mu}{.25}\right) = P(Z < z_0)$$

$z_0 = \frac{5-\mu}{.25}$, $P(Z < z_0) = .02$



From table, $z_0 = -2.05$ (closest)

$$\frac{5-\mu}{.25} = -2.05$$

$$\Rightarrow \mu = 5 + (2.05)(.25) = 5 + .5125 = \boxed{5.5125 \text{ oz}}$$

4.6 THE GAMMA DIST

Def. A r.v. Y has a Gamma dist. with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ ($Y \sim \text{Gamma}(\alpha, \beta)$) if Y has density

$$f(y) = \underbrace{\frac{1}{\beta^{\alpha}\Gamma(\alpha)}}_{\text{Normalizing constant}} y^{\alpha-1} e^{-y/\beta}, \quad y > 0$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$.

Applications

1. Survival times of people or machines.
2. Interarrival times in a queuing system.
3. Incomes.
4. Annual rainfall amounts.
5. χ^2 dist important in statistics.

The Gamma Function

$\Gamma(\alpha)$, $\alpha > 0$, is called the gamma function.

Note that

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{\infty} = -0 + 1 = 1.$$

Also for any $\alpha > 1$,

Integrate by parts:
$\int u\,dv = uv - \int v\,du$
$u = x^{\alpha-1}$ $\quad v = -e^{-x}$
$du = (\alpha-1)x^{\alpha-2}dx$ $\quad dv = e^{-x}dx$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

$$= -x^{\alpha-1}e^{-x}\Big|_{x=0}^{\infty} + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\,dx$$

$$= -0 + 0 + (\alpha-1)\int_0^\infty x^{(\alpha-1)-1} e^{-x}\,dx = (\alpha-1)\Gamma(\alpha-1).$$

Since $\alpha > 1$

Note that $x^a e^{-x} \to 0$ as $x \to \infty$ for any constant a.

Thus, in perticular,

$$\Gamma(1) = 1, \quad \Gamma(2) = (2-1)\Gamma(2-1) = 1, \quad \Gamma(3) = (3-1)\Gamma(3-1) = 2,$$

$$\Gamma(4) = (4-1)\Gamma(4-1) = 3\Gamma(3) = 3\cdot 2! = 3!$$

$$\Gamma(5) = (5-1)\Gamma(5-1) = 4\Gamma(4) = 4\cdot 3! = 4!$$

By induction, $\Gamma(n) = (n-1)!$ for positive integer n.

So the gamma function generalizes the factorial function. Note that $(n-1)!$ is only defined for positive integer n, whereas $\Gamma(\alpha)$ is defined for all real $\alpha > 0$.

See appendix A.1.11 (p. 717)

It can also be shown that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so

$$\Gamma(\tfrac{3}{2}) = (\tfrac{3}{2}-1)\Gamma(\tfrac{3}{2}-1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi},$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{3}{4}\sqrt{\pi}, \quad \Gamma(\tfrac{7}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \tfrac{15}{8}\sqrt{\pi}, \text{ etc.}$$

Properties of the Gamma function

1. $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ for $\alpha > 1$.
2. $\Gamma(n) = (n-1)!$ for positive integer n.

Note: If $Y \sim Gamma(\alpha, \beta)$, then $f(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}, \ y > 0$.

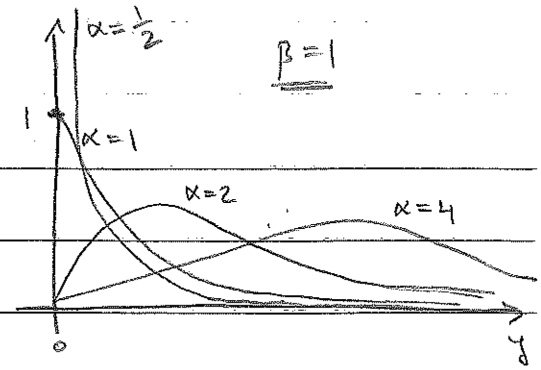
Note that $\quad x = y/\beta \quad y = \beta x \quad dy = \beta dx$

$$\int_0^\infty y^{\alpha-1} e^{-y/\beta} dy = \int_0^\infty (\beta x)^{\alpha-1} e^{-x} \beta dx = \beta^\alpha \int_0^\infty x^{\alpha-1} e^{-x} dx = \beta^\alpha \Gamma(\alpha)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(y) dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \beta^\alpha \Gamma(\alpha) = \underline{1}.$$

controls shape

multiplies distribution by a constant.



Discuss shape (α) and scale (β) parameters.

Draw pictures for $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$.

[For $\alpha > 1$, max at $x = \beta(\alpha - 1)$.]

Two important special cases:

$\underline{\alpha = 1}$: $\quad f(y) = \dfrac{1}{\beta} e^{-y/\beta}, \; y > 0 \qquad$ Exponential Dist (Later...)

$\alpha = \frac{\nu}{2}, \beta = 2$: $\quad$ Chi-square distribution. with ν d.f., $\chi^2(\nu)$

LEMMA. If $Y \sim \text{Gamma}(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, then

$$E\{Y^a\} = \beta^a \frac{\Gamma(\alpha + a)}{\Gamma(\alpha)} \quad \text{if } a > -\alpha.$$

Proof:

$$E\{Y^a\} = \int_{-\infty}^{\infty} y^a f(y) dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^a y^{\alpha - 1} e^{-y/\beta} dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \beta^{\alpha + a} \Gamma(\alpha + a) \underbrace{\int_0^\infty \frac{1}{\beta^{\alpha + a} \Gamma(\alpha + a)} y^{(\alpha + a) - 1} e^{-y/\beta} dy}_{\text{Integral of Gamma}(\alpha + a, \beta) \text{ density as long as } \alpha + a > 0}$$

$$= \frac{\beta^{\alpha + a} \Gamma(\alpha + a)}{\beta^\alpha \Gamma(\alpha)} \quad \text{if } \alpha + a > 0$$

$$= \begin{cases} \beta^a \dfrac{\Gamma(\alpha + a)}{\Gamma(\alpha)} & \text{if } a > -\alpha. \\ \cancel{\infty \quad \text{if } a \le -\alpha}. & \end{cases}$$

(i'm not sure...)

THM If $Y \sim \text{Gamma}(\alpha, \beta)$, $\alpha > 0$, then

$$E(Y) = \alpha\beta,$$

and

$$V(Y) = \alpha\beta^2.$$

Proof. $E(Y) = \beta \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta \cdot \dfrac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta.$

$$E(Y^2) = \beta^2 \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \beta^2 \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta^2 \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \alpha(\alpha+1)\beta^2$$

$$V(Y) = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2.$$

Note: In general, (except for integer α) $F(y) = P(Y \le y)$ is impossible to compute in closed form. The dist function of a Gamma(α, $\beta=1$) r.v. is:

$$F(y) = IG(y) = \frac{1}{\Gamma(\alpha)} \int_0^y x^{\alpha-1} e^{-x}\, dx \quad , \quad y \ge 0$$

is called the incomplete Gamma Function. Its values can be calculated numerically or looked up in tables.

Note: $Y \sim \text{Gamma}(\alpha, \beta)$ $\quad u = x/\beta,\ x = \beta u,\ dx = \beta du$

$$y > 0 \Rightarrow F(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^y x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{y/\beta} (\beta u)^{\alpha-1} e^{-u} \beta du$$

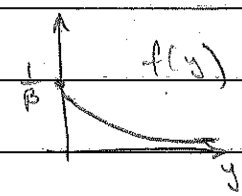
$$= \frac{1}{\Gamma(\alpha)} \int_0^{y/\beta} u^{\alpha-1} e^{-u} du = IG(y/\beta)$$

Book uses Applets

THE EXPONENTIAL DIST

Y has an exponential dist with parameter $\beta > 0$ $[Y \sim Exp(\beta)]$ if Y has density

$$f(y) = \frac{1}{\beta} e^{-y/\beta}, \quad y > 0.$$



Note: $Exp(\beta) \equiv Gamma(\alpha = 1, \beta)$.

Thm. $Y \sim Exp(\beta) \Rightarrow E(Y) = \beta,\ V(Y) = \beta^2$.

Also, $F(y) = \begin{cases} 0, & y \le 0, \\ 1 - e^{-y/\beta}, & y > 0. \end{cases}$

Memoryless Property

Let $a, b > 0$ be constants and let $Y \sim Exp(\beta)$.

Then

$$P(Y > a+b \mid Y > a) = \frac{P(Y > a+b)}{P(Y > a)} = \frac{e^{-(a+b)/\beta}}{e^{-a/\beta}} = e^{-b/\beta} = P(Y > b).$$

"Prob of living at least an additional amount b, given we have lived at least a is indep. of a."

So the exponential distribution can be used to model the survival times of components that do not worsen with age, e.g., transistors. Mention interarrival times as well.

(Interarrival times in P. Process exp dist.)

Ex. If $Y \sim Exp(4)$, find $P(Y \ge 6)$.

$$P(Y \ge 6) = \int_6^\infty \frac{1}{4} e^{-y/4}\,dy = -e^{-y/4}\Big|_6^\infty = e^{-3/2} \approx .22.$$

$$= e^{-6/4}$$

THE CHI-SQUARE (χ^2) DIST

$[Y \sim \chi^2_\nu]$

Defn A r.v. Y has a chi-square dist with ν degrees of freedom
($\nu \geq 1$) if Y has a gamma dist with $\alpha = \nu/2$ and $\beta = 2$.

I.e., $\chi^2_\nu \equiv \text{Gamma}(\alpha = \frac{\nu}{2}, \beta = 2)$.

Very important in statistics. (4343).

Thm If $Y \sim \chi^2_\nu$, then

$$E(Y) = \alpha\beta = \nu \qquad (E(Y) = \text{d.f.})$$

and

$$V(Y) = \alpha\beta^2 = 2\nu.$$

Table 6, pp. 762-3. Mention computers.

4.7 The Beta Probability Dist

Defn A r.v. Y is said to have a beta distribution with parameters α and β, $\alpha > 0$, $\beta > 0$, if Y has p.d.f.

$$f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} & , 0 \le y \le 1, \\ 0 & , \text{elsewhere.} \end{cases}$$

where

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy \overset{!}{=} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Note that we just define the constant to be what we need in order for $\int f = 1$. The fact that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

~~is not obvious~~. requires some calculus [show $\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta)$]

The c.d.f. for the beta r.v. is called the incomplete beta function and is written

$$F(y) = \int_0^y \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)} dt = I_y(\alpha,\beta)$$

Note that $\alpha = 1, \beta = 1$ gives the $U(0,1)$ distn:

$$f(y) = \begin{cases} \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)} y^{1-1}(1-y)^{1-1} = 1, & \text{for } 0 \le y \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Thm If $Y \sim Beta(\alpha, \beta)$, then

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Pf.

$$\mu = E(Y) = \cancel{\frac{1}{B(\alpha+\beta)}} \frac{1}{B(\alpha,\beta)} \int_0^1 y \cdot y^{\alpha-1}(1-y)^{\beta-1}\,dy$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{(\alpha+1)-1}(1-y)^{\beta-1}\,dy \quad \leftarrow \text{kernel of } B(\alpha+1,\beta)$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \underbrace{\left[\frac{1}{B(\alpha+1,\beta)} \int_0^1 y^{(\alpha+1)-1}(1-y)^{\beta-1}\,dy\right]}_{=1}$$

$$= \frac{\dfrac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}} = \frac{\alpha}{\alpha+\beta}$$

Similarly,

$$E(Y^2) = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\Rightarrow V(Y) = E(Y^2) - \mu^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{(\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\cancel{\alpha^2(\alpha+\beta)}\ \alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

what are the mean and variance of the proportion of A in the drug?

$E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{2}{10} = \frac{1}{5}$ (95)

$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{(2)(8)}{10^2 \cdot 11} = \frac{4}{275} \approx (.12)^2$

Ex. Suppose that the proportion of ~~a certain~~ ingredient A in batches of a drug follows a beta distribution with parameters $\alpha = 2$ and $\beta = 8$. Find the probability that a batch of the drug will contain between 10% and 20% ~~ingredient~~ A.

$$f(y) = \frac{1}{B(2,8)} y^{2-1}(1-y)^{8-1}$$

$$\frac{1}{B(2,8)} = \frac{\Gamma(2+8)}{\Gamma(2)\Gamma(8)}$$

$$= 72y(1-y)^7$$

$$= \frac{9!}{1!\,7!} = 9 \cdot 8 = \underline{72}$$

$$P(.10 < Y < .20) = 72\int_{.10}^{.20} y(1-y)^7\,dy$$

One way: Integrate by subst.

Let $u = 1-y$

$y = 1-u$

$dy = -du$

$$= 72\int_{.90}^{.80} (1-u)u^7(-du)$$

$$= 72\int_{.80}^{.90} (1-u)u^7\,du = 72\int_{.8}^{.9} (u^7 - u^8)\,du$$

$$= 72\left[\frac{u^8}{8} - \frac{u^9}{9}\right]_{.8}^{.9}$$

$$= 9u^8\Big|_{.8}^{.9} - 8u^9\Big|_{.8}^{.9}$$

$$= \cancel{72[.0107617 - .0060584]}$$

$$= \cancel{72[.0047032]} = \underline{\underline{.3386}}$$

$$\cancel{72} = 9[(.9)^8 - (.8)^8] - 8[(.9)^9 - (.8)^9]$$

4.9 OTHER EXPECTED VALUES

RECALL:

k^{th} moment: $\mu_k' = E(Y^k)$, $k=1,2,\ldots$

k^{th} central moment: $\mu_k = E[(Y-\mu)^k]$, $k=1,2,\ldots$
(k^{th} moment about the mean)

NOTE: $\mu = \mu_1'$ and $\sigma^2 = \mu_2$

M.G.F.:
$$m(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} \cdot f(y)\,dy$$ for cts Y with pdf $f(y)$.

As before
$$m'(0) = E(Y)$$
$$m''(0) = E(Y^2)$$
and in general
$$m^{(k)}(0) = E(Y^k).$$

Ex. Find the m.g.f. of $Y \sim \text{gamma}(\alpha, \beta)$. Differentiate to get $E(Y)$ and $V(Y)$.

$$m(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{ty} \cdot y^{\alpha-1} e^{-y/\beta}\,dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y\left(\frac{1}{\beta}-t\right)}\,dy$$

$$= \frac{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)}{\beta^\alpha \Gamma(\alpha)} = (1-\beta t)^{-\alpha} \quad \text{for } t < \tfrac{1}{\beta}.$$

Note: Need $t < \frac{1}{\beta}$ for $m(t) < \infty$.

substitute: $x = y\left(\frac{1-\beta t}{\beta}\right)$

$$\frac{1}{\beta} - t = \frac{1-\beta t}{\beta} = \frac{1}{\frac{\beta}{1-\beta t}}$$

Ex. $Y \sim N(\mu, \sigma^2)$ $\qquad f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$, $-\infty < y < \infty$.

$$m(t) = \int_{-\infty}^{\infty} e^{ty} f(y)\, dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ty} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

Let $x = \frac{y-\mu}{\sigma}$ $\qquad y = -\infty \Rightarrow x = -\infty$

$\qquad y = \infty \Rightarrow x = \infty$

$y = \sigma x + \mu$

$dy = \sigma dx$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma x + \mu)} e^{-\frac{1}{2}x^2} \cancel{\sigma}\, dx$$

$$= e^{t\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 + t\sigma x} dx$$

$$-\frac{1}{2}x^2 + t\sigma x = -\frac{1}{2}\left[x^2 - 2t\sigma x + (t\sigma)^2 - t^2\sigma^2\right]$$

$$= -\frac{1}{2}(x - t\sigma)^2 + \frac{1}{2}t^2\sigma^2$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x - t\sigma)^2} dx}_{\text{integral of normal density with mean } t\sigma \text{ and variance } 1.}$$

$$= \boxed{e^{\frac{1}{2}\sigma^2 t^2 + \mu t}}$$

Note that for $Y \sim N(0, 1)$,

$$m(t) = e^{\frac{1}{2}t^2}$$

4.10 Tchebysheff's Thm

For any random variable Y with mean μ and variance σ^2,

$$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \quad \text{or} \quad P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Pf (cts case) [Skip proof – do example next page.]

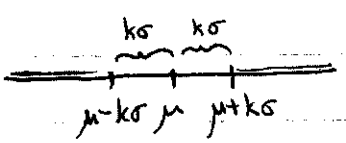
$$\sigma^2 = V(Y) = \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\,dy$$

$$= \int_{-\infty}^{\mu-k\sigma} (y-\mu)^2 f(y)\,dy + \int_{\mu-k\sigma}^{\mu+k\sigma} (y-\mu)^2 f(y)\,dy + \int_{\mu+k\sigma}^{\infty} (y-\mu)^2 f(y)\,dy$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (y-\mu)^2 f(y)\,dy + \int_{\mu+k\sigma}^{\infty} (y-\mu)^2 f(y)\,dy$$

took out middle piece

[min value $y = \mu - k\sigma$] [min value $y = \mu + k\sigma$]



$$\geq k^2\sigma^2 \int_{-\infty}^{\mu-k\sigma} f(y)\,dy + k^2\sigma^2 \int_{\mu+k\sigma}^{\infty} f(y)\,dy$$

$$= k^2\sigma^2 \left(\int_{-\infty}^{\mu-k\sigma} f(y)\,dy + \int_{\mu+k\sigma}^{\infty} f(y)\,dy \right)$$

$$= k^2\sigma^2\, P(|Y-\mu| \geq k\sigma)$$

$$\Rightarrow P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

and of course

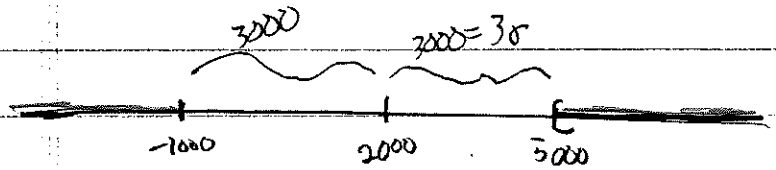
$$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

Ex. The time till failure of a certain type of disc drive follows a gamma dist with parameters $\alpha = 4$ and $\beta = 500$. Give an upper bound on the prob that a disc drive will last at least 5000 hrs.

$$\mu = \alpha\beta = 2000, \quad \sigma = \sqrt{\alpha\beta^2} = 1000 \qquad k = 3$$

$$P(Y \ge 5000) \le P(|Y - 2000| \ge 3000) \le \frac{1}{9} \approx .\overline{111}$$

$$= P(Y > 5000 \ \underbrace{\text{or } Y < -1000}_{\text{not possible for Gamma, } Y > 0})$$



Note: True prob is

$$P(Y > 5000) = P\left(\frac{Y}{500} > \frac{5000}{500}\right) = P(X > 10) \quad \text{where } X \sim \text{Gamma}(\alpha = 4, \beta = 1)$$

$$= 1 - IG(10)$$

$$\approx .0103$$

As usual Tchebysheff is quite conservative. Still useful for theory.