Remember to talk about this in 1st lecture:
- read chapt. 1
- careers in stats.

CHAPTER 1: What is Statistics

EX. Spin a penny 10 times: HTTTTTHTTH

Question: Are H and T equally likely?

p = Prob of T in a single spin

Decide if $p \le \frac{1}{2}$ or $p > \frac{1}{2}$.

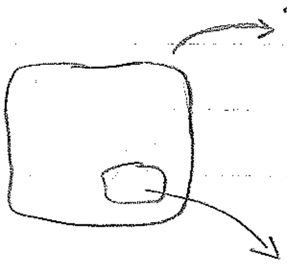
With 10 spins:

# T's	Decision
10	$p > \frac{1}{2}$
9	$p > \frac{1}{2}$ (?)
8	?
7	?

Idea: If $p \le \frac{1}{2}$, then 10 T's very unlikely. Decide $p > \frac{1}{2}$.
But how confident are we of being correct?
What about 9 T's? 8 T's? Would you feel the same about 70 in 100 as about 7 in 10?

A PARADIGM FOR STATISTICS

Two Basic Elements:



Population – large body of data (measurements).
(All possible outcomes of the infinitely many spins that could be made with the penny). (All people w/ heart disease)

Sample – a subset of the pop. that is actually observed.
(The outcome of 10 spins) (A group of 20 h.d. patients)

<u>Objective</u>: Use the sample to <u>infer</u> something about the population.

Statistical Inference —
(studied in 4322, but it relies on the tools of prob studied here).

- Decision making
  - Believe or not that $p = \frac{1}{2}$.
  - Decide which form of ~~cancer~~ h.d. therapy is best
- Estimation
  - Estimate p.
  - Estimate reduction in # ~~cancer~~ h.d. deaths under new treatment
- Prediction?
  - Predict # of H's in next 20 spins. (± ?)
  - Predict survival time of patient.

<u>NOTES. (on stat. inf.)</u>

1. Pop may be <u>concrete</u> (voting preferences of all registered voters in Florida), or
   <u>conceptual</u> (survival times of all present and future prostate cancer patients under a particular treatment).

2. Why only look at sample?

   (a) **Expense**
   (b) Time
   (c) Accuracy (Mention 1990 Census)
   (d) Conceptual population (Cannot give treatment to cancer patients that don't exist yet.)

3. Usually provide a measure of goodness or reliability for the inference.

   (a) How accurate is estimate?
   (b) How confident are we of our decision?

4. Also study how to best design an experiment and how to best use the resulting data to make inferences about the population.

We will study classical, frequentist statistics, which relies on the

RARE EVENT PRINCIPLE: If a sample is obtained which would be very unlikely for a given configuration of the population, then we tend to conclude that the population must not have the given configuration.

Eliminate implausible states of nature (not this, not this, ...)

COIN SPINNING EXAMPLE:

Observe 7 T's in 10 spins.

What's prob 7 or more T's if $p = \frac{1}{2}$?

To understand statistics properly, you must understand probability.

Ancient Vedic saying: "neti, neti, neti, ..., iti."

The Ultimate Reality is beyond human comprehension. The only available route toward understanding The Reality is the way of "not this, not this, ..., this."

CHARACTERIZING A SET OF MEASUREMENTS (DESCRIPTIVE METHODS)

Provides us w/ an assessment or summary of the data.

(Sec 1.2) Graphical Methods

RELATIVE FREQUENCY HISTOGRAM

use their heights as example. $55 \le h < 60$

maybe do a stem-and-leaf plot as well.

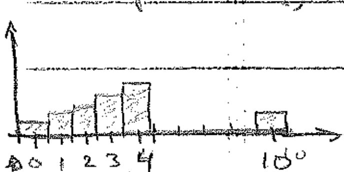
- Line divided into intervals of equal length.
- Bars over intervals have height equal to the relative frequency of the interval (proportion of the measurements which lie in the interval).
- The probability that ~~a~~ a number selected at random from the data will lie in a given interval is proportional to the area of the bars over that interval.
- Often for a population with many values, the histogram is idealized as a frequency curve.

Class Exercise:

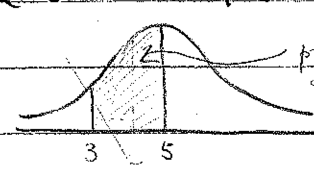
Each toss coin 10 times

(count # of students while they're tossing)

- The height is standardized so that the total area under the curve is 1.
- The prob. that a measurement randomly selected from the pop (~~or the proportion of the pop lying in an interval~~) will be in a given interval is equal to the area under the curve over that interval.



(non-overlapping intervals) [0,1], [1,2], ...



(Sec 1.3) Numerical Methods

Mean of $y_1, \dots, y_n$ is $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

$\bar{y}$ is a measure of location, or center of the y_i's.

$\bar{y}$ = sample mean. μ = pop. mean.

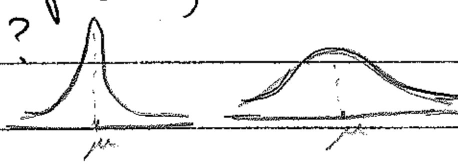
Variance of $y_1, \dots, y_n$ is $\quad s^{*2} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2$.

Std. dev. of $y_1, \dots, y_n$ is $\quad s^* = \sqrt{s^{*2}}$.

Usually s or ~~s^*~~ for sample std dev, σ for pop std dev.

Note: Usually, in stats,
$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2.$$

Comment on s^* as a measure of spread, or variability. How to interpret s^*?
(average squared deviations mean)



EMPIRICAL RULE

For a distribution that is approx. normal (bell-shaped):

$\bar{y} \pm s'$ contains $\approx 68\%$ of the measurements;

$\bar{y} \pm 2s'$ " $\approx 95\%$ " " " ;

$\bar{y} \pm 3s'$ " almost all " " " .

Later will ~~be~~ look at Chebyshev's Thm.

Ex

Scores on a test have $\mu = 64$, $\sigma = 10$

$\Rightarrow$ ~ 68% of scores b/ween 54 & 74

95% " " " 44 & 84

almost all " " " 34 & 94

CH. 2

WHAT IS PROBABILITY? (and why study it?)

"Probability does not exist." (outside the mind.)

Circular

Subjective Def'n – Subjective measure of the likelihood of occurrence of a future or unobservable event.

"What is the probability that Miami will win the Super Bowl this year?"

Frequentist Def'n – The long run relative frequency of an event in a sequence of repeated independent trials. (Explain relative frequency: prop. of trials that result in that event)

"What is the probability of heads when a balanced coin is tossed?"

emphasize difference b/ween these two: can actually carry out an experiment to find P(heads)

Random or stochastic mechanism – Unpredictable outcome in a single trial, but predictable long run behavior. Relative freq's of outcomes approach fixed values. These fixed values are the probabilities of the outcomes.

Ex1. Coin tossing – Cannot predict with certainty whether next toss will result in H. But know that in a sufficiently long sequence of tosses the proportion of tosses resulting in heads will be as close to ½ as we care to specify.

OVER

We need a more rigorous def^n of prob.
We will take an axiomatic approach.

Axiomatic Approach — Theory of prob. built up from a set of axioms which specify how prob's ought to behave mathematically. Does not depend on how you choose to interpret or employ probability.

~~Prob. could be done in a void, without reference to intuition, but would lose much of its richness. Our intuition usually suggests what ought to be true, or what questions we should ask.~~

§2.3 EVENTS IN THE SAMPLE SPACE

(~~St~~istical)
Experiment – the process by which an obs. is made. Outcome cannot be predicted with certainty. Ex. Toss a coin twice. Play an NFL season.

Sample point – one of the individual possible outcomes of the experiment.

Sample space – collection of all sample points. ← Defining the sample space "correctly" is often the most important step.
(The "universal set".)

notion / discrete sample space.

Ex 2 Toss two coins (a nickel and a penny). Sample points might be labeled
HH, HT, TH, TT.
Sample space,
$$S = \{HH, HT, TH, TT\}.$$

Event – a collection of sample points (a set in the sample space).

Simple event – an event consisting of only one sample point.

Empty event ($\emptyset$) – Event containing no sample points.

Example 2. Toss 3 coins.

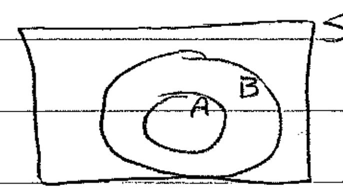
$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Some events:

$$A = \{\text{obs. one } H\} = \{HTT, THT, TTH\}$$
$$B = \{\text{obs. two } H\text{'s}\} = \{HHT, HTH, THH\}$$
$$C = \{\text{obs. at least one } H\} = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$$
$$D = \{\text{obs. at most one } H\} = \{HTT, THT, TTH, TTT\}$$
$$E = \{\text{obs. an odd no. of } H\text{'s}\} = \{HHH, HTT, THT, TTH\}$$

SET NOTATION AND VENN DIAGRAMS

1. $A \subseteq B$: A is contained in B; A is a subset of B; A "implies" B

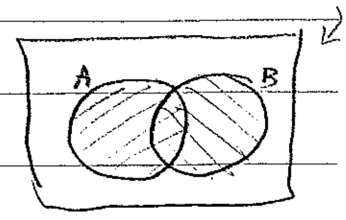


Venn Diag.

Example: In example 2,

$A \subset C$, $A \subset D$, $A \subset E$, $B \subset C$, etc.

2. $A \cup B$: union of A and B := all sample points contained in either A or B (or both); A "or" B



Example: In example 2, $B \cup E = C$

May also write AB for this

3. $A \cap B$: intersection of A and B := all sample points shared by A and B, i.e., the points they have in common.

A "and" B ; both events occur (simultaneously).

Example: In example 2, $D \cap \bar{E} = \{HTT, THT, TTH\} = A$

~~$C \cap D = \{HTT, THT, TTH\}$~~

or use logic

4. A and B are mutually exclusive (disjoint) events if $A \cap B = \phi$.
(cannot occur simultaneously)

Example: In example 2, B and E are mutually exclusive. (also A & B).

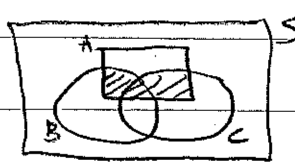
5. $\bar{A}$: compliment of A := all sample points not in A;
"not" A.

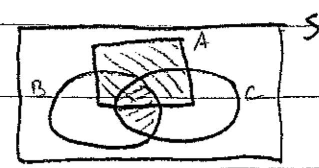
Example 2. $\bar{C} = \{\text{no H's}\} = \{TTT\}$.

ALGEBRA OF EVENTS (Interpret in logical terms)

Distributive Laws : (1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(1) 

(2) 

DeMorgan's Laws : (1) $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

(2) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

They should convince themselves of this!

Logic. Venn Diagrams?

§2.4 Axioms of Probability

Def Probability ~~Model consists of~~

- Sample space S, associated with an experiment.
- A is any event in S $(A \subset S)$.
- Assign a finite real value, $P(A)$ (the probability of A), to each event A in such a way that the following axioms hold:

Axiom 1 : $P(A) \geq 0 \quad \forall$ events A (nonnegative)

Axiom 2 : $P(S) = 1$

Axiom 3 : (Countable additivity) If $A_1, A_2, \ldots$ are pairwise mutually exclusive events, i.e., $A_i \cap A_j = \phi \ \forall i,j, i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

~~Just take this to include finite additivity~~?

Important consequences of the axioms

1. Let $A_1 = A_2 = \cdots = \phi$. Then $\bigcup_{i=1}^{\infty} A_i = \phi$ and $A_i \cap A_j = \phi$ $\forall i,j$. Thus

$$P(\phi) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\phi) \quad [\text{Axiom } 3]$$

Since $P(\phi) \geq 0$ [Axiom 1], this implies that $P(\phi) = 0$.

2. Let $A_1, \ldots, A_n$ be pairwise mutually exclusive.
Let $A_{n+1} = A_{n+2} = \cdots = \phi$. Then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i \quad \text{and still } A_i \cap A_j = \phi \ \forall\, i \neq j,$$

so

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(\phi) = \sum_{i=1}^{n} P(A_i)$$

This establishes finite additivity: For $A_1, \ldots, A_n$ pairwise mutually exc

$$P\left(\bigcup_{1}^{n} A_i\right) = \sum_{1}^{n} P(A_i)$$

3. Since $A \cup \bar{A} = S$ and $A \cap \bar{A} = \phi$, we have

$$\begin{aligned} 1 &= P(S) && \text{(Axiom 1)} \\ &= P(A \cup \bar{A}) \\ &= P(A) + P(\bar{A}) && \text{(finite additivity)} \end{aligned}$$

$$\Rightarrow P(\bar{A}) = 1 - P(A).$$

In this chapter we will always work with a discrete sample space: S contains either a finite or countable number of sample points. (Countable: ~~In this case~~ can be put into 1-1 corresp. w/ the integers.)

$S = \{E_1, E_2, \ldots\}$.

omit this

~~(i) The collection of events can always be taken to be all subsets of S (the power set of S).~~

this is ok!

~~(ii) If we assign a nonnegative-probability to each simple event in such a way that~~

$$\sum_{i=1}^{\infty} P(E_i) = 1,$$

~~then we have a legitimate probability model. The probability of any event A is determined by adding up the probabilities of all the simple events contained in A.~~

NOTE: The axioms only tell us what properties the probabilities must satisfy. They do not tell us exactly what the probabilities must be.

Ex. Tossing 3 coins. The "equally likely" model assigns equal probs to each of the sample points. Alt. model: H twice as likely as T.

Sample Point (Single Events)	Probability: Equally Likely Model	Probability: Alt. Model $\left(P(H) = \frac{2}{3}\right)$
HHH	$1/8$	$8/27$
HHT	$1/8$	$4/27$
HTH	$1/8$	$4/27$
HTT	$1/8$	$2/27$
THH	$1/8$	$4/27$
THT	$1/8$	$2/27$
TTH	$1/8$	$2/27$
TTT	$1/8$	$1/27$
	1	1

$$A = \{\text{even \# of H's}\} = \{HHT, HTH, THH, TTT\}$$

$$P(A) = P(\{HHT\}) + P(\{HTH\}) + P(\{THH\}) + P(\{TTT\})$$

$$= \begin{cases} \frac{1}{2}, & \text{under equally likely model,} \\ 13/27, & \text{under alternative model } \left(P(H) = 2/3\right) \end{cases}$$

§2.5 Calculating the Prob of an Event: The Sample Point Method (Event A)

~~In a discrete model, it suffices to assign (or be given) the prob. of each simple event.~~

~~Note that the sum of the prob of all the simple events is 1.~~

~~Sample point method obtains prob of an event by adding the probs of all the sample points in the event. This is just exploiting countable or finite additivity.~~

~~Things are more complicated for uncountable sample spaces.~~

1. Define the exp.
2. List the sample points of the exp.
3. Assign reasonable probs to the sample points.
   (Make sure $P(E_i) \geq 0$ and $\Sigma P(E_i) = 1$.)
4. Define A as a collection of sample points. ($A = \bigcup_{i=1}^{n} E_i$)
5. Find P(A) by summing the probs of the sample points in A.

$$P(A) = \sum_{i=1}^{n} P(E_i).$$

Example. Select 2 of 5 systems at random for testing. Two of the 5 are actually defective.

A = {at least one of the two selected is defective}

B = {neither of the two selected is defective}

C = {both selected are defective}

Label systems D_1, D_2, G_1, G_2, G_3. Note that we are sampling without replacement. Two different ways to do this problem.

1st way

Sample point: (1st system selected, 2nd system selected)

All permutations of 2 of the 5 items: $P^5_2 = \frac{5!}{3!} = 20$

Sample Space:

D_1D_2	D_2D_1	D_2G_1	G_1D_2	G_1G_2	G_2G_1
D_1G_1	G_1D_1	D_2G_2	G_2D_2	G_1G_3	G_3G_1
D_1G_2	G_2D_1	D_2G_3	G_3D_2	G_2G_3	G_3G_2
D_1G_3	G_3D_1				

C | A | B

20 sample points, all equally likely, so all simple events receive probability $\frac{1}{20}$.

$$P(A) = P(D_1D_2) + P(D_2D_1) + \cdots + P(G_3D_2) = \frac{14}{20} = \frac{7}{10}$$

$$P(B) = \frac{6}{20} = 1 - P(A) \text{ since } B = \overline{A}.$$

$$P(C) = \frac{2}{20} = \frac{1}{10}$$

NOTE: Since we are sampling without replacement, order isn't important in determining which systems we end up with. It's important that the events ~~we're~~ of interest ~~in~~ also do not involve order.

2nd ~~Another~~ Way

Sample point: (two systems drawn, order not important)

Sample space: D_1D_2 D_2G_2 D_1G_1 D_2G_3 D_1G_2 G_1G_2 D_1G_3 G_1G_3 D_2G_1 G_2G_3

All combinations of 2 of the 5 objects. $C^5_2 = \frac{5!}{2!\,3!} = 10$

~~Just need to consider all subsets of 2 of the 5 items~~

Each simple event has prob. $\frac{1}{10}$

$P(A) = \frac{7}{10}$

$P(B) = \frac{3}{10} = 1 - P(A)$

$P(C) = \frac{1}{10}$

CHANGE THE PROBLEM

Suppose now that we sample with replacement (could get same system twice).

Sample point: (1st system drawn, second system drawn) ordered pair

Sample space:

D_1D_1	D_2D_1	G_1D_1	G_2D_1	G_3D_1
D_1D_2	D_2D_2	G_1D_2	G_2D_2	G_3D_2
D_1G_1	D_2G_1	G_1G_1	G_2G_1	G_3G_1
D_1G_2	D_2G_2	G_1G_2	G_2G_2	G_3G_2
D_1G_3	D_2G_3	G_1G_3	G_2G_3	G_3G_3

All simple events have prob $\frac{1}{25}$.

$P(A) = \frac{16}{25}$, $P(B) = \frac{9}{25}$, $P(C) = \frac{4}{25}$.

<u>2nd way</u> (with replacement, order not important)

<u>sample point</u>: (2 systems drawn, order not important)

$\underline{S}$:

D_1D_1	D_1D_2	D_1G_1	D_1G_2	D_1G_3
D_2D_2	D_2G_1	D_2G_2	D_2G_3	
G_1G_1	G_1G_2	G_1G_3		
G_2G_2	G_2G_3			
G_3G_3				

C   A   B

$$\#S = \underset{\uparrow \text{ as before}}{10} + \underset{\uparrow \text{ repeats}}{5} = 15$$

(but these are not equally likely!)

~~$P(A) = \frac{9}{15}$~~ e.g. $P(D_1D_1) = \frac{1}{25}$, $P(D_1D_2) = \frac{2}{25}$

$$\Rightarrow P(C) = \frac{1}{25} + \frac{1}{25} + \frac{2}{25} = \frac{4}{25}$$

etc.

(START HERE)

→ Could also do this 2^{nd} way (order not important) but now sample points are not all equally likely! [See above – which may be skipped.] (yes, skip it.)

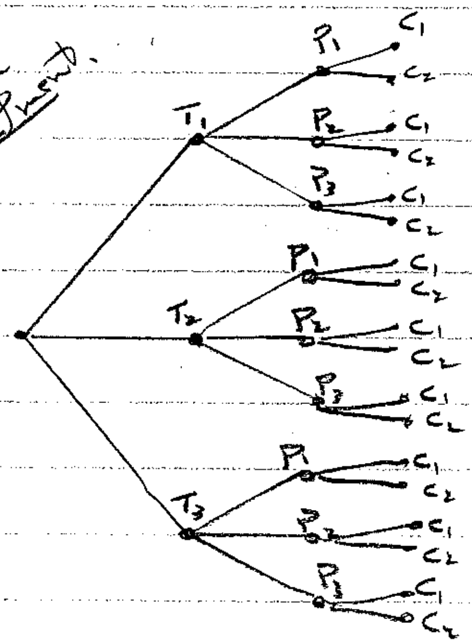
→ Note also that in sampling with replacement, the outcome of 2^{nd} draw is independent of the outcome of 1^{st} draw. ~~This is NOT true when sampling~~ (Like rolling two dice, or tossing two coins.)
This is NOT true when sampling w/o replacement.

§2.6 Counting Rules

Ex. Experiment to investigate the effect of pressure (3 levels), temperature (3 levels), and type of catalyst (2 levels) on yield.
How many different treatments are possible?

Note: A treatment is a particular combination of levels of the three factors.

Imagine constructing a particular treatment.



$3 \times 3 \times 2 = 18$ different treatments.

This is an example of the mn Rule. Basically this is a multiplication rule for counting the number of ways in which a sequential procedure can be carried out.

We will use this idea a lot in this chapter.

Ex. How many four letter "words" are possible using a 26 letter alphabet if

(a) letters can be repeated within a word, i.e., bbbb is a word?

$$26^4 = 456{,}976$$

(b) No repeated letters are allowed?

$$26 \cdot 25 \cdot 24 \cdot 23 = 358{,}800$$

Part (b) is an example of counting permutations.

Def. A permutation is an ordered arrangement of distinct objects.

The # of permutations of n distinct objects is $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

Ex. How many 4 letter "words" can be written using the letters a, b, c, d with no repeats?

$$4! = 4 \cdot 3 \cdot 2 \cdot 1.$$

The # of permutations of n distinct objects taken r at a time is

$$P^n_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Note that $P^n_n = \frac{n!}{0!} = n!$

Ex. # of 4 letter "words" possible using 26 letters with no repeats is

$$P^{26}_4 = \frac{26!}{22!} = 26 \cdot 25 \cdot 24 \cdot 23.$$

Ex. 20 club members. Choose a pres, vice pres., and treasurer. How many ways can this be done?

$$P^{20}_{3} = \frac{20!}{17!} = 20 \cdot 19 \cdot 18.$$

Ex. (Old Exam Prob) Given a list of 10 words and 15 definitions. Each word must be matched with a definition with no repeats.

(i) What is the probability of getting all 10 correct if just guessing?

Matching 10 of the definitions with the 10 words creates a 10-permutation of the 15 definitions. Note that order is important.
The number of such permutations is

def ... def

w_1 ... w_{10}

$$\#(S) = P^{15}_{10} = \frac{15!}{5!} = 15 \cdot 14 \cdot 13 \cdots 7 \cdot 6 \approx 1.0897 \times 10^{10}$$

$$P(\text{all correct}) = \frac{1}{P^{15}_{10}} = \frac{1}{15!/5!} = \frac{5!}{15!} \approx 9.18 \times 10^{-11}$$

(ii) What is the probability of getting $\underbrace{\text{all but one correct}}_{B}$?

$$\#(B) = 10 \cdot 5 = 50 \Rightarrow P(B) = \frac{10 \cdot 5}{15!/5!} = \frac{50}{15!/5!}$$

w_1 w_2 ... w_i ... w_{10}

Imagine all slots correctly filled. Now replace w_1 with one of the 5 remaining defs. Can choose any slot in 10 ways

$\Rightarrow \#(B) = 5 \times 10$.

Gets harder after this. Why?

Birthday Problem

Ignore leap years. 365 distinct possible birthdays.
Consider all equally likely.

In a class of 30, what is the prob. that at least two people share a birthday? Naive answer: $\frac{30}{365} = .082$

Order people alphabetically, or by SSN, or by seat, or some other way. $\underset{P_1}{\sqcup}\ \underset{P_2}{\sqcup} \cdots \underset{P_{30}}{\sqcup}$

# of diff. possible configurations of birthdays is

$$\#(S) = 365^{30} \approx 7.39 \times 10^{76}.$$

So the prob. of each configuration under equally likely model is $1/365^{30}$.

$A = \{\text{at least two share a birthday}\}$, $\#(A) = ?$
$\bar{A} = \{\text{no two share a birthday}\}$

$P(A) = 1 - P(\bar{A})$.

$$\#(\bar{A}) = \underbrace{365 \cdot 364 \cdots 336}_{30 \text{ factors}} = \frac{365!}{335!} = P^{365}_{30} \approx 2.17 \times 10^{76}$$

So

$$P(\bar{A}) = \frac{365!/335!}{(365)^{30}} \approx .2937$$

$$\Rightarrow P(A) = 1 - P(\bar{A}) = 1 - \frac{365!/335!}{(365)^{30}} \approx .7063$$

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= #(S) = Cardinality of sample space

$\text{Prob} = \frac{\text{Od}}{1+\text{Od}}$

Num_people	Num_poss	Prob_match	Odds_match		
20	1.76e+51	0.411	0.70		
21	6.42e+53	0.444	0.80		
22	2.34e+56	0.476	0.91		
23	8.56e+58	0.507	1.03		0.51
24	3.12e+61	0.538	1.17		
25	1.14e+64	0.569	1.32		
26	4.16e+66	0.598	1.49		
27	1.52e+69	0.627	1.68		
28	5.54e+71	0.654	1.89		
29	2.02e+74	0.681	2.13		
30	7.39e+76	0.706	2.41		0.71
31	2.69e+79	0.730	2.71	.73	
32	9.84e+81	0.753	3.05	.75	
33	3.59e+84	0.775	3.44	.77	
34	1.31e+87	0.795	3.89	.80	
35	4.78e+89	0.814	4.39	.81	
36	1.74e+92	0.832	4.96	.83	
37	6.38e+94	0.849	5.61	.85	
38	2.32e+97	0.864	6.36	.86	
39	8.50e+99	0.878	7.21	.88	
40	3.10e+102	0.891	8.19		0.89
41	1.13e+105	0.903	9.33		
42	4.13e+107	0.914	10.63		
43	1.50e+110	0.924	12.14		
44	5.50e+112	0.933	13.90		
45	2.00e+115	0.941	15.94		
46	7.33e+117	0.948	18.32		
47	2.67e+120	0.955	21.11		
48	9.77e+122	0.961	24.38		
49	3.56e+125	0.966	28.22		
50	1.30e+128	0.970	32.75		0.97
51	4.75e+130	0.974	38.11		
52	1.73e+133	0.978	44.46		
53	6.33e+135	0.981	52.02		
54	2.31e+138	0.984	61.02		
55	8.43e+140	0.986	71.79		
56	3.07e+143	0.988	84.71		
57	1.12e+146	0.990	100.24		
58	4.10e+148	0.992	118.98		
59	1.49e+151	0.993	141.64		
60	5.46e+153	0.994	169.15		0.99
61	1.99e+156	0.995	202.62		
62	7.28e+158	0.996	243.47		
63	2.65e+161	0.997	293.50		
64	9.69e+163	0.997	354.93		
65	3.54e+166	0.998	430.61		
66	1.29e+169	0.998	524.13		
67	4.71e+171	0.998	640.04		
68	1.72e+174	0.999	784.17		
69	6.28e+176	0.999	963.94		
70	2.29e+179	0.999	1188.88		

Ask Class to find # @ least one match

Ex: Choose P & VP from 4 available people.

$$\#S = \frac{4!}{2!} = 12 \quad \rightarrow \begin{Bmatrix} AB & BA & CA & DA \\ AC & BC & CB & DB \\ AD & BD & CD & DC \end{Bmatrix}$$

Ex: Choose Committee of 2 from 4 available.

$$\#S = \begin{Bmatrix} AB & \cancel{BA} & \cancel{CA} & \cancel{DA} \\ AC & BC & \cancel{CB} & \cancel{DB} \\ AD & BD & CD & \cancel{DC} \end{Bmatrix} = 6 = \frac{4!/2!}{2!}$$

Combinations

The # of subsets (combinations) of r objects which can be formed from a collection of n distinct objects is:

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof:

$$P_r^n = \frac{n!}{(n-r)!} = r!\, C_r^n .$$

Note: ~~Ex~~

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Binomial Coefficient.

$$\binom{n}{n} = \binom{n}{0} = 1$$

$$\binom{n}{n-1} = \binom{n}{1} = n$$

Ex. Three committee members to be selected at random from a group of 8 candidates, 5 women and 3 men. What is prob that none of the 3 men are selected?

Total # committees $= \binom{8}{3} = 56$

committees w/ no men $= \binom{5}{3}\binom{3}{0} = 10 \cdot 1 = 10$

$P(\text{no men}) = \frac{10}{56} = \frac{5}{28} = .1786$

Ex (ctd) What is prob that exactly one man is selected?

committees w/ one man $= \binom{3}{1} \cdot \binom{5}{2} = 3 \cdot \binom{5}{2} = 3 \cdot 10 = 30$

$P(\text{one man}) = \frac{30}{56} = \frac{15}{28} = .5357$

Ex. 5 Card Poker.

Standard card deck: $\underbrace{\{2, 3, \ldots, 10, Q, J, K, A\}}_{\text{(13 denominations or kinds)}} \times \underbrace{\{H, D, C, S\}}_{\text{(4 suits)}}$

for a total of 52 cards.

$\# S = \binom{52}{5}$

<u>Find:</u>

(a) $P(4 \text{ aces}) = \binom{4}{4}\binom{48}{1} / \#S$

(b) $P(\text{exactly } 2 \text{ aces}) = \binom{4}{2}\binom{48}{3} / \#S$

(c) $P(\text{a pair}) = P(\text{two of same kind}) = \binom{13}{1}\binom{4}{2}\binom{50}{3} / \#S$

(d) $P(\text{exactly one pair}) = \binom{13}{1}\binom{4}{2}\binom{48}{3} / \#S$?

No, this gives more pairs

Should be: $\binom{12}{3} 4^3$

(e) $P(\text{full house}) = ?$

Full house: 3 of same kind and 2 of same kind.

Reason as follows:

$\binom{4}{3}$ ways to choose 3 cards from a given kind.

$\binom{4}{2}$ " " " 2 " " " " "

$\binom{13}{2}$ " " " 2 kinds from the 13 available.

$\Rightarrow P(\text{full house}) = \binom{4}{3}\binom{4}{2}\binom{13}{2} / \#S = ?$

No! (Subtle error.)

Imagine ~~for~~ ex: $\left.\begin{matrix} 2_D\, 2_S\, 2_C\, Q_H\, Q_C \\ Q_D\, Q_S\, Q_C\, 2_H\, 2_C \end{matrix}\right\}$ different!

Thus the order in choosing kinds matters.

$\Rightarrow P(FH) = \binom{4}{3}\binom{4}{2} P^{13}_{2} / \#S \simeq .0014$.

Note that

$$(x_1+x_2+\cdots+x_k)^n = \sum_{\{n_1+\cdots+n_k=n\}} \binom{n}{n_1\, n_2 \cdots n_k} x_1^{n_1}\cdots x_k^{n_k}$$

<u>MULTINOMIAL COEFFICIENT</u> (generalization of Binomial coefficients or combinations)

$$\binom{n}{n_1\; n_2\; \cdots\; n_k} = \#$$ of ways of dividing n distinct objects into k gps of sizes $n_1, n_2, \ldots, n_k$, where $n_i \geq 0\ \forall i$ and $n = n_1+n_2+\cdots+n_k$.

Note that

$$\binom{n}{n_1\; n_2\; \cdots\; n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \cdots = \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

<u>NOTE</u>: $\binom{n}{k} = \binom{n}{k\;\; n-k}$

<u>Ex.</u> Philosophy faculty to be ~~assigned~~ divided at random ~~to~~ into 3 committees consisting of 6, 4, and 2 members, resp. In how many ways can this be done?

$$\binom{12}{6\;4\;2} = \frac{12!}{6!\,4!\,2!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot \cancel{8}^{2}\cdot 7}{(\cancel{4\cdot 3\cdot 2\cdot 1})(2\cdot 1)} = 13{,}860$$

<u>Ex (ctd)</u> If 3 of the faculty are women, what is prob that both the persons on the 3rd committee are women?

~~A = {both faculty on 3rd committee are women}~~

$$\#(A) = \binom{3}{2}\binom{10}{6\;\;4} = 3\cdot 210 = 630$$

$$P(A) = \frac{630}{13{,}860} = .0455$$

Ex (ctd) What is prob that all 3 women are assigned to the committee #6

$B = \{$ all 3 women assigned to committee #6 $\}$

$$\#(B) = \binom{3}{3}\binom{9}{3\ 4\ 2} = (1)\left(\frac{9!}{3!\,4!\,2!}\right) = 1260$$

$$P(B) = \frac{1260}{13{,}860} = .0909$$

Summary of Counting Rules

ways to fill r slots from n available distinct objects is:

sample	order matters (slots distinguishable)	order ~~NOT~~ matters (slots indistinguishable)
w/o replacement	P^n_r	C^n_r
with replacement	n^r $\underbrace{n\ n\ \ldots\ n}_{r}$	$\binom{n-1+r}{r}$ tricky!

§2.7 CONDITIONAL PROB

Given that an event B has occurred, what is the probability that the event A has also occurred? $P(A|B)$.

Ex. Roll 2 dice. Given that sum is 6, what is conditional prob. that at least one one of the two dice showed a 2?

$B = \{\text{sum is } 6\}$, $A = \{\text{at least one of the dice showed a } 2\}$

want $P(A|B)$.

36 possible outcomes for ~~the~~ expt: (roll 1, roll 2)

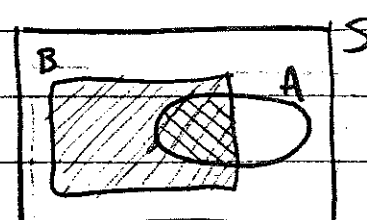
$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$A = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (1,2), (3,2), (4,2), (5,2), (6,2)\}$

Seems reasonable that $P(A|B) = 2/5$.

The idea is to reduce the sample space to the given event B, and then see what proportion of the prob of B also belongs to A. Thus we define

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Ex. In the dice example above:

$$\left.\begin{array}{l} P(B) = 5/36 \\ A \cap B = \{(2,4), (4,2)\} \\ P(A \cap B) = 2/36 \end{array}\right\} P(A|B) = \frac{2/36}{5/36} = \frac{2}{5}.$$

f_B = freq of B in N trials

$f_{A\cap B}$ = freq of $A \cap B$ in N trials

$$\frac{f_{A\cap B}}{f_B} = \frac{f_{A\cap B}/N}{f_B/N} \to \frac{P(A \cap B)}{P(B)} \text{ as } N \to \infty$$

Do not confuse $P(A|B)$ with $P(B|A)$., they are different in general!!

Ex. (ctd) In the example,

$$P(A) = 11/36$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

Ex. At a certain college,

25% of students fail math

15% " " " chemistry

10% " " " both.

A student is to be selected at random.

Find $P(\underbrace{\text{student failed chemistry}}_{A} \mid \underbrace{\text{student failed math}}_{B})$

$P(B) = .25 \qquad P(A \cap B) = .10$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.10}{.25} = 2/5 = .40$$

Compare $P(A|B) = .40$ with $P(A) = .15$

Ex:

~~Exercise #2.50, p.49~~ . $D = \{\text{customer dissatisfied}\}$

$A = \{\text{job done by plumber } A\}$

A survey of consumers in city showed 10% diss. w/ plumbing jobs. Half of complaints were about plumber A who does 40% of plumbing jobs in town.

Given $P(D) = .10$, $P(A) = .40$, and

$$P(A|D) = \frac{1}{2} = .50$$ (make sure they get this.)

Given that customer is dissatisfied, prob of having had A as plumber is 0.5.

Find prob consumer will be diss. given that A comes to provide service

(b) $$P(D|A) = \frac{P(A \cap D)}{P(A)} = \frac{.05}{.40} = \frac{1}{8} = .125$$

Want $P(A \cap D)$. ~~Note that~~

This first

Find prop of consumers who are dissatisfied and are serviced by A

(a) Now, $$P(A|D) = \frac{P(A \cap D)}{P(D)} \Rightarrow P(A \cap D) = P(D)P(A|D)$$

$$= (.10)\left(\tfrac{1}{2}\right) = .05$$

Explain that this is intuitively obvious.

Note that here plumber A has more (50%) than his "fair share" (40%) of the dissatisfied customers.

Thus it is not surprising that

$$P(D|A) = .125 > P(D) = .10 .$$

Find prob that consumer will be satisfied given A provides service

(c) $$P(\bar{D}|A) = 1 - P(D|A) = 1 - \frac{1}{8} = \frac{7}{8} = .875.$$

Note: $$\underbrace{P(\bar{A}|B) = 1 - P(A|B)}$$

Conditioning event remains the same.

INDEPENDENCE

DEF Two events A and B are *independent* if $P(A \cap B) = P(A)P(B)$.

Why? If A and B are independent and $P(B) \neq 0$,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

Similarly, if $P(A) \neq 0$,

$$P(B|A) = P(B).$$

So if any of these happens $\Rightarrow$ indep.

> **NOTE:** $P(A \cap B)$
>
> INDEPENDENT $\neq$ MUTUALLY EXCLUSIVE
>
> Explain. $P(A)\cdot P(B)$ $\qquad$ 0

Ex. Tossing two fair coins. Assume all sample pts in $S = \{HH, HT, TH, TT\}$ are equally likely. Let

$A = \{$first coin is H$\}$

$B = \{$second coin is H$\}$

$C = \{$at least one coin is a H$\}$

$D = \{$both coins yield same outcome$\}$ ~~$= \{$even # of H's$\}$~~

$E = \{1^{st}$ coin is T$\}$

$A = \{HH, HT\}$, $B = \{HH, TH\}$, $C = \{HH, HT, TH\}$, $D = \{HH, TT\}$

$E = \{TH, TT\}$.

$A \cap B = \{HH\}$, $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$

$\Rightarrow$ A and B are independent. (but not mutually exc.)

(But A & E are mutually exc. & not indep. $P(A \cap E) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} = P(A)P(E)$)

$A \cap C = \{HH, HT\} = A$ $\qquad$ $P(A \cap C) = \frac{1}{2} \neq \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = P(A) \cdot P(C)$

do this last!

$\Rightarrow$ A and C are dependent (not independent).

NOTE. Events $A, B,$ and C are pairwise independent if

$$* \begin{cases} P(A\cap B) = P(A)P(B) \\ P(A\cap C) = P(A)\cdot P(C) \\ P(B\cap C) = P(B)\cdot P(C) \end{cases}$$

$A, B,$ and C are independent if in addition to $*$,

$$P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C).$$

Ex. In the last example, let

$D = \{$both coins yield the same outcome$\}$

$= \{HH, TT\}$

$P(D) = \frac{1}{2}$

Check that $A, B,$ and D are pairwise independent but are not independent. ~~Makes sense since $A\cap B \Rightarrow D$~~

Usually independence is suggested by the setting of a problem.

$$P(A\cap B) = P(A)\cdot P(B) \quad ✓$$

$$P(A\cap D) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2} = P(A)\cdot P(D)$$

$$P(B\cap D) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2} = P(B)\cdot P(D)$$

But: $P(A\cap B\cap D) = \frac{1}{4} \neq \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = P(A)\cdot P(B)\cdot P(D)$.

§2.8 Two Laws of Prob

1. Multiplicative Law

$$P(A\cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Proof: From def. of cond. prob.

Ex: Draw 2 cards randomly from a 52-card deck. what is prob they are both aces?

$A_1 = \{\text{ace on } 1^{st} \text{ draw}\}$, $A_2 = \{\text{ace on } 2^{nd} \text{ draw}\}$

$$P(A_1 \cap A_2) = P(A_2|A_1)P(A_1) = \frac{3}{51}\cdot\frac{4}{52} =$$

$$= \frac{\binom{4}{2}}{\binom{52}{2}} \quad (\text{a longer computation})$$

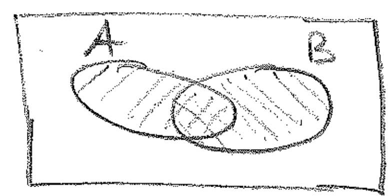
General Case:

$$P(A_1\cap A_2 \cap \dots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)\dots P(A_k|A_1\cap\dots\cap A_{k-1})$$

2. Additive Law

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

Proof: Venn Diagram.



Extension:

$$P(A\cup B\cup C) = \cancel{P(A)+P(B)}\, P(A\cup B) + P(C) - P((A\cup B)\cap C)$$

$$\vdots$$

$$= P(A)+P(B)+P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Extends to more events, "Inclusion–Exclusion Formula".

Note: $A \perp\!\!\!\perp B \iff A \perp\!\!\!\perp \bar{B} \iff \bar{A} \perp\!\!\!\perp B \iff \bar{A} \perp\!\!\!\perp \bar{B}$.

__Ex.__ Two students take an exam independently. First student will fail with prob .4, second fails with prob .2.
What is prob at least one fails?

A_1 = 1st student fails, A_2 = 2nd student fails

$$P(\text{at least one fails}) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
$$= P(A_1) + P(A_2) - P(A_1) \cdot P(A_2)$$
$$= .4 + 2 - (.4)(.2) = .4 + .2 - .08 = .52$$

__OR__

$$P(\text{at least one fails}) = 1 - P(\text{neither fails}) = 1 - P(\bar{A}_1 \cap \bar{A}_2)$$
$$= 1 - P(\bar{A}_1) \cdot P(\bar{A}_2)$$
$$= 1 - (.6)(.8) = .52$$

__Ex.__ $\exists$ A and B events such that
$$P(A) = .6, \quad P(B) = .5, \text{ and } P(A \cup B) = .8.$$
Are A and B independent?

$$P(A \cap B) \overset{?}{=} P(A) \cdot P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B) = .6 + .5 - .8 = .3$$
$$P(A) \cdot P(B) = (.6)(.5) = .3$$

Yes, A and B are independent.

Another way: $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = .2$
$P(\bar{A})P(\bar{B}) = (1-.6)(1-.5) = (.4)(.5) = .2$

So $\bar{A} \perp\!\!\!\perp \bar{B}$, which implies $A \perp\!\!\!\perp B$.

2.9 EVENT COMPOSITION METHOD (for finding probs)

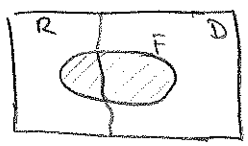
Idea: Express a given event as a composition involving unions and/or intersections of other events.

Ex: 40% of voters R, 60% D. Among R, 70% in favor of a bond issue, whereas 80% of D favor same issue. If a voter randomly selected, what is prob he/she favors bond issue?

F = "favors bond issue".

Know: $P(R) = .4$, $P(D) = .6$, $P(F|R) = .7$
$P(F|D) = .8$

$$F = (F \cap R) \cup (F \cap D) \quad \text{mut. excl.}$$



$$\Rightarrow P(F) = P(F \cap R) + P(F \cap D) \quad \text{, Axiom 3.}$$

$$= P(F|R)P(R) + P(F|D)P(D)$$

$$= (.7)(.4) + (.8)(.6) = \underline{0.76}.$$

Ex. System consists of 3 subsystems connected serially.

1st subsystem has 1 component of type A.

2nd ——— 3 components —— B in parallel.

3rd ——— 2 ——— C in parallel.

Type A components have reliability .95.

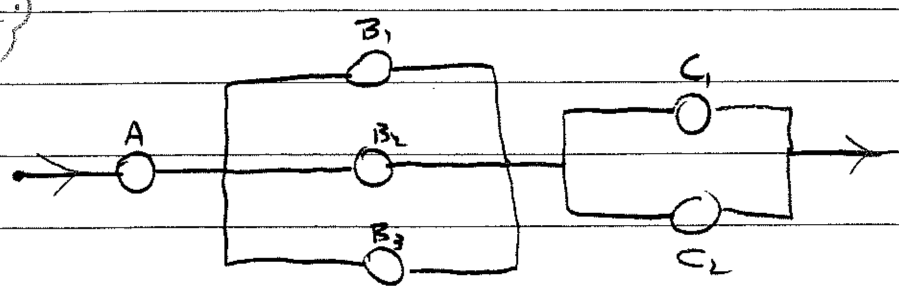
— B ——— .80

— C ——— .90.

Components operate or fail independently.

What is reliability of the system?

works if current can get thru.



$$P(\text{system works}) = P(A \cap (B_1 \cup B_2 \cup B_3) \cap (C_1 \cup C_2))$$

$$= P(A)\,P(B_1 \cup B_2 \cup B_3)\,P(C_1 \cup C_2)$$

$$= P(A)\left[1 - P(\bar{B}_1 \bar{B}_2 \bar{B}_3)\right]\left[1 - P(\bar{C}_1 \bar{C}_2)\right] \quad \text{De Morgan}$$

$$= P(A)\left[1 - P(\bar{B}_1)P(\bar{B}_2)P(\bar{B}_3)\right]\left[1 - P(\bar{C}_1)P(\bar{C}_2)\right]$$

$$= (.95)\left[1 - (.20)^3\right]\left[1 - (.10)^2\right]$$

$$= (.95)(1 - .008)(1 - .01)$$

$$= (.95)(.992)(.99) \approx .933$$

✓ Ex (~~#2.71, p.60~~) [This one really uses Law of Tot. Prob.]

Let

$M = \{$customer sees Magazine ad$\}$

$T = \{$ " " TV " $\}$

$A = M \cup T = \{$customer sees at least one ad$\}$

$B = \{$customer buys product$\}$.

Want: $P(B)$.

Given: $P(M) = \frac{1}{50}$, $P(T) = \frac{1}{5}$, $P(M \cap T) = \frac{1}{100}$

$P(B|A) = \frac{1}{3}$, $P(B|\bar{A}) = \frac{1}{10}$.

Sol'n:

$$B = (B \cap A) \cup (B \cap \bar{A}) \quad \text{and} \quad (B \cap A) \cap (B \cap \bar{A}) = \phi$$

$$\Rightarrow P(B) = P(B \cap A) + P(B \cap \bar{A})$$

$$= P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$

But $P(A) = P(M \cup T) = P(M) + P(T) - P(M \cap T)$

$$= \frac{1}{50} + \frac{1}{5} - \frac{1}{100} = \frac{21}{100},$$

$$P(\bar{A}) = 1 - P(A) = \frac{79}{100},$$

so

$$P(B) = \left(\frac{1}{3}\right)\left(\frac{21}{100}\right) + \left(\frac{1}{10}\right)\left(\frac{79}{100}\right)$$

$$= \frac{7}{100} + \frac{79}{1000} = \underline{\underline{\frac{149}{1000}}} = \underline{\underline{0.149}}$$

2.10 LAW OF TOTAL PROBABILITY

If $S = B_1 \cup B_2 \cup \cdots \cup B_k$, where $P(B_i) > 0$, $i = 1, \ldots, k$, and $B_i \cap B_j = \phi$ for $i \neq j$, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i).$$

$B_1, \ldots, B_k$ are mutually exclusive and exhaustive

Ex. Factory has 3 machines.

Machine 1 produces 50% of factory's output,
" 2 " 30% " " " ,
" 3 " 20% " " " .

Also,

Machine 1 produces 3% defectives,
" 2 " 4% " ,
" 3 " 5% " .

Item selected at random from days output. What is prob. the item will be defective?

Let

$A = \{\text{item selected defective}\}$

$B_i = \{\text{item produced by } i^{\text{th}} \text{ machine}\}$, $i = 1, 2, 3$.

Then

$$S = B_1 \cup B_2 \cup B_3 \quad \text{and} \quad B_i \cap B_j = \phi, \ i \neq j.$$

So by Law of Tot. Prob.,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \\ &= (.03)(.50) + (.04)(.30) + (.05)(.20) = \underline{\underline{.037}} \end{aligned}$$

✓ Ex. (ctd) Given that item is defective, what is the cond. prob. that it was produced by machine 1?

$$P(B_1|A) = \frac{P(A\cap B_1)}{P(A)}$$

In the last part, we computed $P(A)$ using the Law of Tot. Prob. as

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$
$$= .037$$

But

$$P(A\cap B_1) = P(A|B_1)P(B_1) = (.03)(.50) = .015$$

$$\Rightarrow P(B_1|A) = \frac{.015}{.037} = \underline{\underline{.4054}}$$

The last example actually demonstrates Bayes' Rule:

If $S = B_1 \cup \cdots \cup B_k$, $P(B_i) > 0$, $i = 1, \ldots, k$, and $B_i \cap B_j = \phi$ for $i \neq j$, then for any event $A \ni P(A) > 0$,

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}.$$

Perhaps go back and do $P(A|B)$ in magazine ex.

This follows easily from Law of Tot. Prob., since

$$P(B_j|A) = \frac{P(A\cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}$$

← Mult Law

← Law of Tot. Prob

Go back and do $P(A|B)$ in exercise #2.71.

✓ Ex. Test for HIV. "False positive rate" is 5%, "false negative rate" is 4%. 1% of the population has the virus. A randomly selected individual tests positive for HIV. What is the conditional probability that the person has HIV.

Let: $A = \{\text{has HIV}\}$

$B = \{\text{tests positive}\}$.

Want: $P(A|B)$.

Given: $P(B|\bar{A}) = .05$

$P(\bar{B}|A) = .04 \Rightarrow P(B|A) = .96$

$P(A) = .01 \Rightarrow P(\bar{A}) = .99$

Sol'n:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Rule

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

$$= \frac{(.96)(.01)}{(.96)(.01) + (.05)(.99)} = \underline{\underline{.1624}}$$

If false +ve rate reduced to 1%, $P(A|B) \simeq 0.5$

$\frac{1}{10}\%$, $P(A|B) \simeq 0.9$

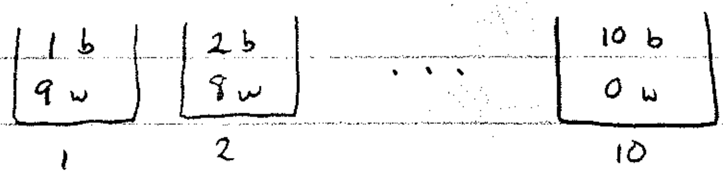
#5 on handout

✓ <u>Ex</u>. 10 urns, 10 balls in each urn.

k^{th} urn has k black balls and $10-k$ white balls, $k=1,\dots,10$.

An urn is selected at random and a ball is selected at random from that urn.

Given that the ball selected is black, what is the (conditional) probability that it was taken from the k^{th} urn?



Let: $B = \{\text{ball chosen is black}\}$

$U_i = \{i^{th} \text{ urn chosen}\}$

Want: $P(U_k \mid B)$

Given: $P(U_i) = \frac{1}{10}$, $i = 1,\dots,10$

$P(B \mid U_i) = \frac{i}{10}$

<u>Soln</u>:

$$P(U_k \mid B) = \frac{P(U_k \cap B)}{P(B)}$$

$$P(U_k \cap B) = P(B \mid U_k)P(U_k) = \frac{k}{10}\cdot\frac{1}{10} = \frac{k}{100}$$

$$P(B) = \sum_{i=1}^{10} P(B \mid U_i)P(U_i) = \sum_{i=1}^{10}\left(\frac{i}{10}\right)\left(\frac{1}{10}\right)$$

$$= \frac{1}{100}\sum_{i=1}^{10} i = \frac{1}{100}\left(\frac{10(10+1)}{2}\right) = \frac{55}{100}$$

$$\Rightarrow P(U_k \mid B) = \frac{k/100}{55/100} = \boxed{\frac{k}{55}}.$$

#4 on handout.

Ex. (Feller, p.141) 5 men out of 100 are colorblind and

25 women " " 10,000 " "

A person is chosen at random and found to be colorblind. What is the (conditional) prob. of his being male?

Assume males and females occur in equal numbers.

$M = \{\text{person is male}\}$, $F = \bar{M} = \{\text{person is female}\}$

$C = \{\text{person is colorblind}\}$

$P(M) = P(F) = .50$

$P(C|M) = \frac{5}{100}$

$P(C|F) = \frac{25}{10{,}000}$

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{P(C|M)\,P(M)}{P(C|M)\,P(M) + P(C|F)\,P(F)}$$

$$= \frac{\left(\frac{5}{100}\right)\left(\frac{1}{2}\right)}{\left(\frac{5}{100}\right)\left(\frac{1}{2}\right) + \left(\frac{25}{10{,}000}\right)\left(\frac{1}{2}\right)} = \frac{500}{500 + 25}$$

$$= \frac{500}{525} = \underline{\underline{\frac{20}{21}}} = \underline{\underline{.9524}}$$

Same as asking what proportion of colorblind people are male.

Prob #13 — #7 on handout

Ex (Feller, p. 141)

1^{st} die has 4 red and 2 white faces.
2^{nd} die " 2 " " 4 " ".

A coin is flipped once.
If H, game continues by throwing 1^{st} die alone.
If T, " " " " 2^{nd} die ".

(a) What is prob of red at any throw?

$$P(R_k) = P(R_k|H)P(H) + P(R_k|T)P(T)$$
$$= \left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right) = \underline{\underline{\tfrac{1}{2}}}$$

(b) If the first two throws resulted in red, what is the probability of red at the third throw?

$$P(R_3|R_1 \cap R_2) = \frac{P(R_1 \cap R_2 \cap R_3)}{P(R_1 \cap R_2)}$$

$$P(R_1 \cap R_2 \cap R_3) = P(R_1R_2R_3|H)P(H) + P(R_1R_2R_3|T)P(T)$$
$$= \left(\tfrac{2}{3}\right)^3\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}\right)^3\left(\tfrac{1}{2}\right)$$
$$= \left(\tfrac{8}{27}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{27}\right)\left(\tfrac{1}{2}\right) = \frac{9}{54} = \frac{1}{6}$$

$$P(R_1R_2) = P(R_1R_2|H)P(H) + P(R_1R_2|T)P(T)$$
$$= \left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}\right)^2\left(\tfrac{1}{2}\right)$$
$$= \left(\tfrac{4}{9}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{9}\right)\left(\tfrac{1}{2}\right) = 5/18$$

$$P(R_3|R_1R_2) = \frac{1/6}{5/18} = \frac{1}{\cancel{6}_1} \cdot \frac{\cancel{18}^3}{5} = \boxed{3/5}$$

Ex (ctd) (Feller, prob #13, p.141)

(c) If red turns up on all of the first n throws, what is the probability that the first die is being used?

$$P(H/R_1\cdots R_n) = \frac{P(H\cap R_1\cdots R_n)}{P(R_1\cdots R_n)}$$

$$P(H\cap R_1\cdots R_n) = P(R_1\cdots R_n|H)P(H)$$

$$= \left(\tfrac{2}{3}\right)^n\cdot\tfrac{1}{2}$$

$$P(R_1\cdots R_n) = P(R_1\cdots R_n|H)P(H) + P(R_1\cdots R_n|T)P(T)$$

$$= \left(\tfrac{2}{3}\right)^n\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}\right)^n\left(\tfrac{1}{2}\right)$$

$$\Rightarrow P(H/R_1\cdots R_n) = \frac{\left(\frac{2}{3}\right)^n\cancel{\left(\frac{1}{2}\right)}}{\left(\frac{2}{3}\right)^n\cancel{\left(\frac{1}{2}\right)} + \left(\frac{1}{3}\right)^n\cancel{\left(\frac{1}{2}\right)}}$$

$$= \frac{2^n/3^n}{(2^n+1)/3^n} = \frac{2^n}{2^n+1}.$$

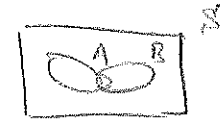
$$P(H) = \tfrac{1}{2}$$
$$P(H|R_1) = \tfrac{2}{3}$$
$$P(H|R_1R_2) = \tfrac{4}{5}$$
$$P(H|R_1R_2R_3) = \tfrac{8}{9}$$
$$\vdots$$

CHAPTERS 1 & 2 — REVIEW

Events:

De Morgan's Laws: $\overline{\bigcup_i^n A_i} = \bigcap_i^n \overline{A}_i$

$\overline{\bigcap_i^n A_i} = \bigcup_i^n \overline{A}_i$



Probs: $0 \le P(A) \le 1$, $P(\overline{A}) = 1 - P(A)$, $P\left(\bigcup_i^n A_i\right) = \sum_i^n P(A_i)$, $A_i \cap A_j = \phi$.

Sample Point Method: $P(A) = \dfrac{\#\text{ sample pts in } A}{\#\text{ sample pts in } S}$ ← if all points equally likely

Counting Rules: Questions: (1) w/ or w/o replacement
(2) order matters or not.

P_r^n = # ways of picking r distinguishable objs from n and arranging them in order.

$= C_r^n \, r! = \dfrac{n!}{r!(n-r)!} r!$

$\binom{n}{n_1 \ldots n_k} = \left(\dfrac{n!}{n_1! \ldots n_k!}\right)$ = # ways of partitioning n distinguishable obj's into groups of size $n_1, \ldots, n_k$.

Cond. Prob: $P(A|B) = \dfrac{P(AB)}{P(B)}$

Indep: $P(AB) = P(A).P(B)$

Additive Law: $P(A \cup B) = P(A) + P(B) - P(AB)$.

Law of total prob & Baye's Rule: $S = B_1 \cup \ldots \cup B_k$, $B_i \cap B_j = \phi$

$P(B_j|A) = \dfrac{P(AB_j)}{P(A)}$

$= \dfrac{P(A|B_j)P(B_j)}{\sum_1^k P(A|B_i)P(B_i)}$.