Approach to Solve Constant Coefficients Equation

To solve any $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ for $a_n, \ldots, a_0$

, we will follow these steps:

1. Assume is a solution
2. Plug y into (must take derivatives:)
3. Cancel out of the equation
4. Solve algebraically for all possible $(m_1, m_2, \ldots, m_n)$
5. Solution is $y =$

* Note: there will be special rules for " / " roots.

This solution approach turns an into an

to solve. But we need "n" solutions for a

fundamental solution, so we will need to have "n" .

ex $2y'' - 5y' - 3y = 0$ ⊛

Assume $y =$. Then $y' =$ and $y'' =$

The ODE is $0 =$

$= [\qquad]$ (factor out e^{mx}).

The auxiliary equation is $0 = \qquad =$

So, $m_1 =$ and $m_2 =$. The general solution is:

$y = c_1 \qquad + c_2 \qquad \longrightarrow \boxed{y = \qquad}$

ex $y'' - 10y' + 25y = 0$

If $y =$, $y' =$, $y'' =$, then we get

$0 = \qquad = [\qquad]$

$\rightarrow 0 = \qquad = (\qquad)(\qquad)$. i.e., $m_1 = m_2 =$

We don't have 2 roots, so we can't write $y = c_1 e^{\,x} + c_2 e^{\sim}$.

Instead, the solution will be: $y =$

Repeated Roots *

If some root " " is repeated " " times, then we use fundamental solutions e , e , e , ..., e

exp $y''' + 3y'' - 4y = 0$

$y =$, $y' =$, $y'' =$, $y''' =$

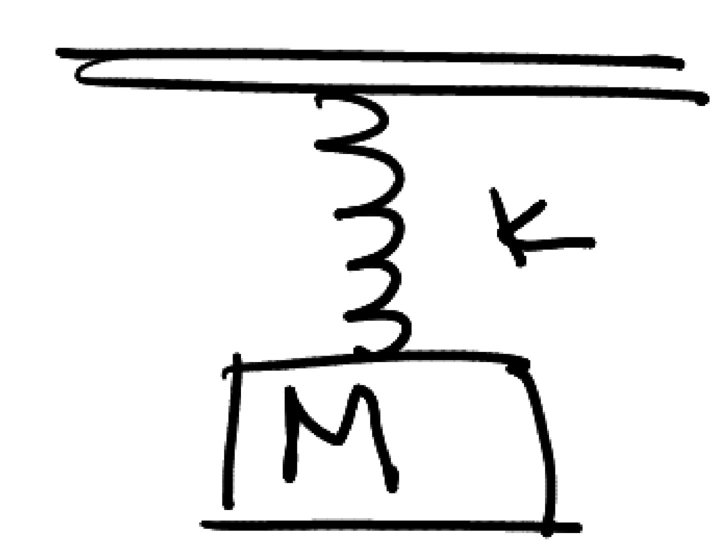
$0 =$ $=$ [ ]

$\rightarrow 0 =$ $=($ $)($ $)=($ $)($ $)($ $)$

So, $m =$ and $m =$ (repeated).

The solution is $y =$

exp For $x(t)$ the displacement of a spring (with mass $M =$ and spring coefficent $k =$) from equilibrium, the spring equation is $\frac{1}{16}\frac{d^2x}{dt^2} + 4x = 0$.



Or, $\begin{cases} x'' + \quad x = 0 \\ x(0) = \frac{2}{3}, \ x'(0) = -\frac{4}{3} \end{cases}$ (the IVP for this spring)

If $x =$, then $x' =$ and $x'' =$

$0 = x'' + 64x =$ $= [\qquad]$

$\rightarrow 0 =$ $\rightarrow m^2 =$, or $m =$

This gives us $y =$ which is hard to interpret.

Imaginary Roots*

Note: $e^{i\theta} =$

If $m = \pm \beta i$, use solutions and

If $m = \alpha \pm \beta i$, use solutions and

This works because complex roots always appear in pairs, and we can rewrite $e^{\alpha + i\beta} =$ to apply the imaginary root.

Imaginary and Repeated Roots*

If complex $m = \alpha \pm \beta i$ is repeated "p" times, use solutions

$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ \ldots,\ x^{p-1}e^{\alpha x}\cos\beta x$

$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ \ldots,\ x^{p-1}e^{\alpha x}\sin\beta x$

ex For $m =$ $\rightarrow$ $x = c_1 \qquad + c_2$

$x' =$

$x(0) = \frac{2}{3} = \qquad = \qquad \rightarrow c_1 =$

$x'(0) = -\frac{4}{3} = \qquad =$

Or, $-\frac{4}{3} = \qquad \rightarrow c_2 =$

The solution is $x(t) =$

ex The equation for a damped spring is $\frac{d^2x}{dt^2} + \quad \frac{dx}{dt} + \quad x = 0.$

By taking $2\lambda =$ and $\omega^2 =$, we can rearrange this to the form $x'' + \quad x' + \quad x = 0.$

If $x =$, $x' =$, and $x'' =$, the ODE is

$0 = \qquad = [\qquad]$

The auxiliary equation is $0 =$

which has solutions $m =$

Simplifying, $m =$

$=$

If $\lambda = 0$, there is no and $m =$ (like before)

Otherwise, we have three options:

① If $\lambda^2 - \omega^2 < 0$, $m =$ for $\beta =$

This is a complex case $\longrightarrow$ $e^{\alpha x} \sin \beta x$, $e^{\alpha x} \cos \beta x$

② If $\lambda^2 - \omega^2 > 0$, $\sqrt{}$ is a number and we have 2 real roots $m =$ $\longrightarrow$

③ If $\lambda^2 - \omega^2 = 0$, then we have root $m =$

This is a case $\longrightarrow$

ex ①

$$x'' + 2x' + 10x = 0$$

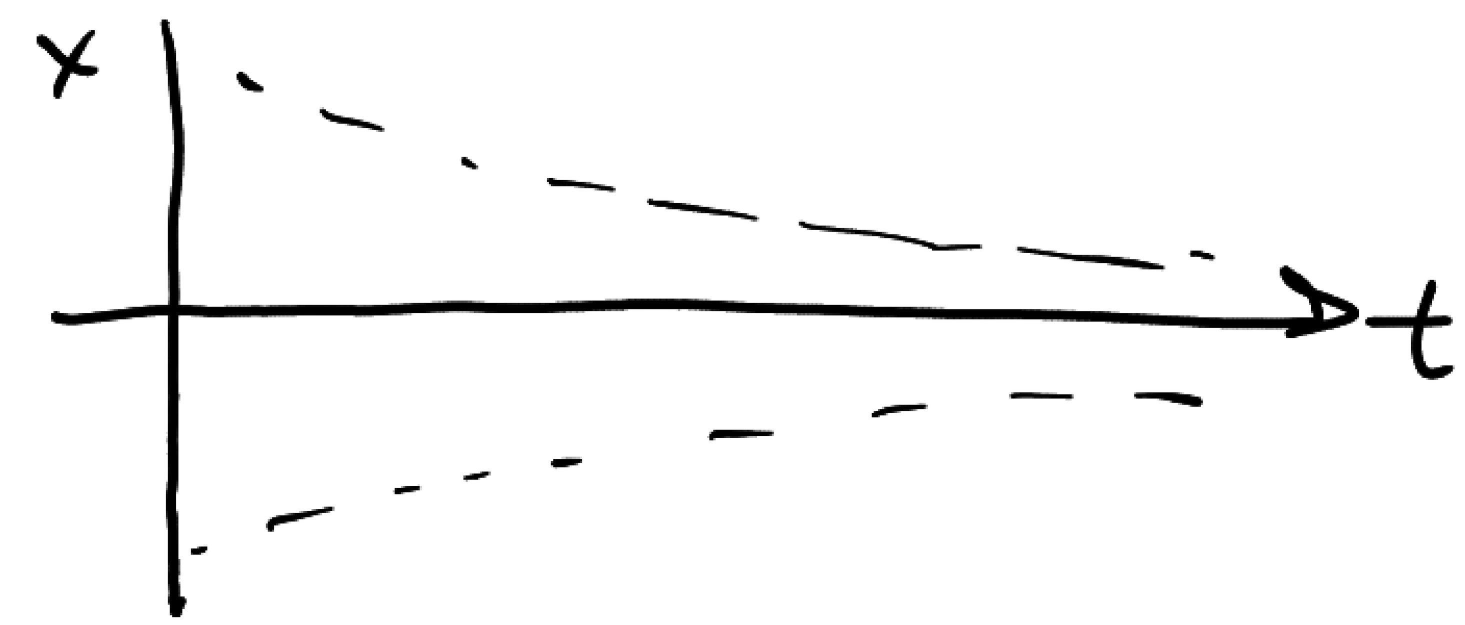
$0 =$ $= [$ $]$

The equation $0 =$ has quadratic roots,

$m =$

That is, $m = \alpha + \beta i$ for $\alpha =$ and $\beta =$, so

$x = e \quad [c_1 \quad + c_2 \quad]$



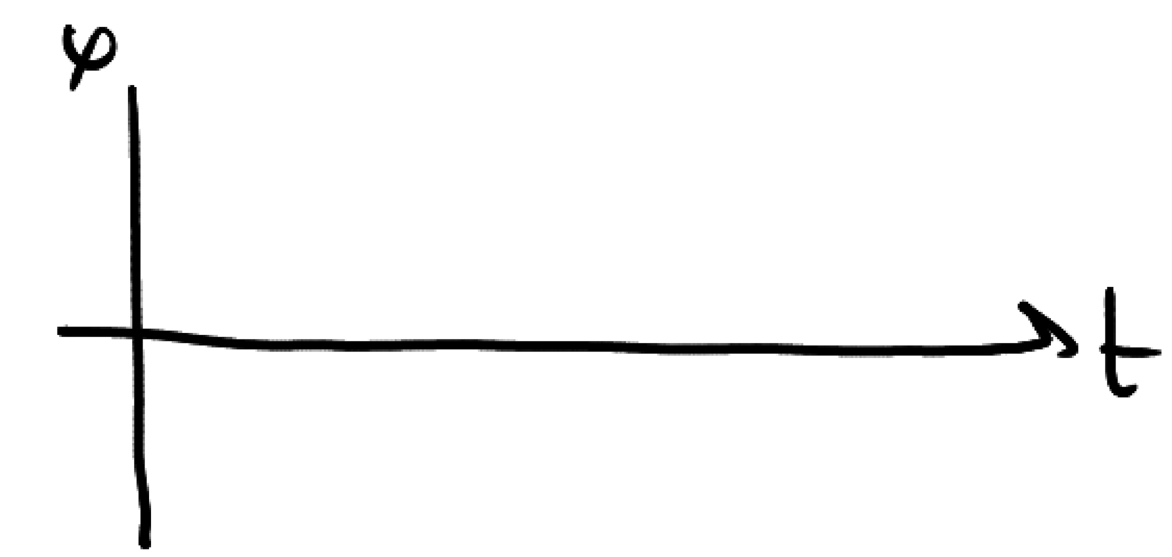
ex 2

$x'' + 5x' + 4x = 0$

$0 = \qquad = [\qquad]$

Then $0 = \qquad = (\qquad)(\qquad) \rightarrow m =$

The solution is $x = c_1 \qquad + c_2$



ex 3

$x'' + 8x' + 16x = 0$

$0 = \qquad = [\qquad]$

Then $0 = \qquad = (\qquad)(\qquad) \rightarrow m =$ (repeated)

The solution is $x = c_1 \qquad + c_2$