

Proposition. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ have radius of convergence $R = R(f) > 0$.

Then,

a) $g(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ has radius of convergence $R(g) \geq R(f)$. In fact,
 $R(g) = R(f) = R$.

b) $f \in A(B(z_0, R))$ and $f' = g$

c) f is infinitely differentiable on $B(z_0, R)$ and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}$$

d) $a_n = \frac{f^{(n)}(z_0)}{n!}$

Proof: We note that together a) and b) inductively imply c).

a) Wolog $z_0 = 0$. We have $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R = R(f) > 0$.

Let $|z| = r < \mathbf{r} < R$.

It is known that $\lim_{n \rightarrow \infty} n \left(\frac{r}{\mathbf{r}} \right)^n = 0$. Hence,

$$\exists N \ni \left[n \geq N \Rightarrow n \left(\frac{r}{\mathbf{r}} \right)^n < r \right] \Rightarrow \left[n \geq N \Rightarrow n r^{n-1} < \mathbf{r}^n \right]$$

Hence, $n \geq N \Rightarrow |n a_n z^{n-1}| < |a_n| \mathbf{r}^n$ which implies $R(g) \geq R(f)$.

On the other hand, let $f^* = z * g$. Then, $R(f^*) \geq \min(R(z), R(g)) = R(g)$. But, since $\limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n}$. We have therefore $R(f) = R(f^*) \geq R(g)$. Hence, $R(g) = R(f) = R$.

b) WoLog $z_0 = 0$. We need to show that $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = g(z)$ for $z \in B(0, R)$. Equivalently, we need to show $\lim_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} - g(z) \right| = 0$ for $z \in B(0, R)$. Let $|z| = r < \mathbf{r} < R$. Choose $h \ni |h| < \mathbf{d} = \frac{1}{2}(\mathbf{r} - r)$. Note that $r < \mathbf{r} - 2\mathbf{d}$. We have now

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=1}^{\infty} a_n \left[\frac{(z+h)^n - z^n}{h} - nz^{n-1} \right] \quad (1)$$

Rewriting the term in brackets we have

$$\frac{(z+h)^n - z^n}{h} - nz^{n-1} = \sum_{j=1}^n \binom{n}{j} z^{n-j} h^{j-1} - nz^{n-1} = \sum_{j=2}^n \binom{n}{j} z^{n-j} h^{j-1}$$

Hence,

$$\begin{aligned} \left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| &= \left| \sum_{j=2}^n \binom{n}{j} z^{n-j} h^{j-1} \right| \\ &\leq |h| \sum_{j=2}^n \binom{n}{j} |z|^{n-j} |h|^{j-2} \\ &\leq |h| \sum_{j=2}^n \binom{n}{j} (\mathbf{r} - 2\mathbf{d})^{n-j} \mathbf{d}^{j-2} \end{aligned}$$

Since

$$\begin{aligned} \binom{n}{j} &= \frac{n!}{(n-j)!j!} = \frac{n(n-1)(n-2)!}{[(n-2)-(j-2)]!(1 \cdot 2 \cdot 3 \cdots j)} \\ &\leq \frac{n(n-1)}{2} \frac{(n-2)!}{[(n-2)-(j-2)]!(j-2)!} \\ &= \frac{n(n-1)}{2} \binom{n-2}{j-2} \end{aligned}$$

we have

$$\begin{aligned} \left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| &\leq |h| \frac{n(n-1)}{2} \sum_{j=2}^n \binom{n-2}{j-2} (\mathbf{r} - 2\mathbf{d})^{n-j} \mathbf{d}^{j-2} \\ &= |h| \frac{n(n-1)}{2} (\mathbf{r} - \mathbf{d})^{n-2} \end{aligned} \quad (2)$$

Since, $\mathbf{r} < R$, the series $\sum_{n=0}^{\infty} |a_n| \mathbf{r}^n$ converges, which implies $\lim_{n \rightarrow \infty} |a_n| \mathbf{r}^n = 0$, which implies

$\exists M \ni \forall n \geq 0 |a_n| \mathbf{r}^n \leq M$. Hence, $\forall n \geq 0 |a_n| \leq \frac{M}{\mathbf{r}^n}$. Combining this with (1) and (2)

we have

$$\begin{aligned} \left| \frac{f(z+h) - f(z)}{h} - g(z) \right| &\leq |h| \sum_{n=1}^{\infty} \frac{M n(n-1)}{\mathbf{r}^n} (\mathbf{r} - \mathbf{d})^{n-2} \\ &= |h| \frac{M}{\mathbf{r}^2} \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \left(\frac{\mathbf{r} - \mathbf{d}}{\mathbf{r}} \right)^{n-2} \end{aligned}$$

Since this latter series converges, say to K , then $\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \leq |h| \frac{M}{\mathbf{r}^2} K$. Hence,

as $h \rightarrow 0$ we have $\lim_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} - g(z) \right| = 0$.

d) The formula for a_k follows from c) by evaluating $f^{(k)}(z)$ at $z = z_0$.