Definition. Let G be a region and let $\mathrm{a} = a + bi = (a,b) \in G$. A function $g : G \to \mathbb{R}$ is differentiable at a if there exist constants A and B and functions $e_1, e_2 : G \to \mathbb{R}$ which are continuous at a with $e_1(a,b) = e_2(a,b) = 0$ such that

$$g(x,y) = g(a,b) + (x-a)\{A + e_1(x,y)\} + (y-b)\{B + e_2(x,y)\}$$

If g is differentiable at a, we will define $g_x(a,b) = A$ and $g_y(a,b) = B$.

Proposition. Let G be a region and let $\mathrm{a} = a + bi = (a,b) \in G$. Suppose that $u, v : G \to \mathbb{R}$ such that u, v are (continuously) differentiable at a and that u, v satisfy the Cauchy-Riemann equations at a. Then, $f = u + iv$ is (continuously) differentiable at a.

Proof. Since u, v are differentiable at a, there exist functions $e_r : G \to \mathbb{R},\ r = 1, 2, 3, 4,$ which are continuous at a with $e_r(a,b) = 0$, $r = 1, 2, 3, 4,$ such that

$$u(x,y) = u(a,b) + (x-a)\{u_x(a,b) + e_1(x,y)\} + (y-b)\{u_y(a,b) + e_2(x,y)\} \quad (1)$$
$$v(x,y) = v(a,b) + (x-a)\{v_x(a,b) + e_3(x,y)\} + (y-b)\{v_y(a,b) + e_4(x,y)\} \quad (2)$$

If in (1) we replace $u_y(a,b) = -v_x(a,b) = i \cdot i \cdot v_x(a,b)$ and in (2) we replace $v_y(a,b) = u_x(a,b)$ and then multiply (2) by i and add the result to (1) we obtain (after a rearrangement)

$$f(z) - f(\mathrm{a}) = (z - \mathrm{a})[u_x(a,b) + i\, v_x(a,b) + w(z)]$$

where

$$w(z) = \frac{x-a}{z-\mathrm{a}}\{e_1(x,y) + ie_3(x,y)\} + \frac{y-b}{z-\mathrm{a}}\{e_2(x,y) + ie_4(x,y)\}.$$

Since $|w(z)| \le |e_1(x,y)| + |e_2(x,y)| + |e_3(x,y)| + |e_4(x,y)|$, then

$$\lim_{z \to a} \frac{f(z) - f(\mathrm{a})}{z - \mathrm{a}} = \lim_{z \to a} u_x(a,b) + i\, v_x(a,b) + w(z) = u_x(a,b) + i\, v_x(a,b)$$

since $\lim_{z \to a} w(z) = 0$.