3.3

1. $\{z \mid \operatorname{Re} z < 0,\ |\operatorname{Im} z| < \pi\} \xrightarrow{e^z} \mathbb{D} \setminus (-1, 0]$

3a) $\sin(-z) = -\sin z$, $\sin \bar{z} = \overline{\sin z}$, $\sin(\pi + z) = -\sin z$, $\sin(z + 2\pi) = \sin z$

$(\sin z)' = \cos z \Rightarrow (\sin z)' = 0$ for $z = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$

$\Rightarrow$ need only consider $HVS_s = \{z \mid \operatorname{Im} z > 0,\ |\operatorname{Re} z| < \frac{\pi}{2}\}$

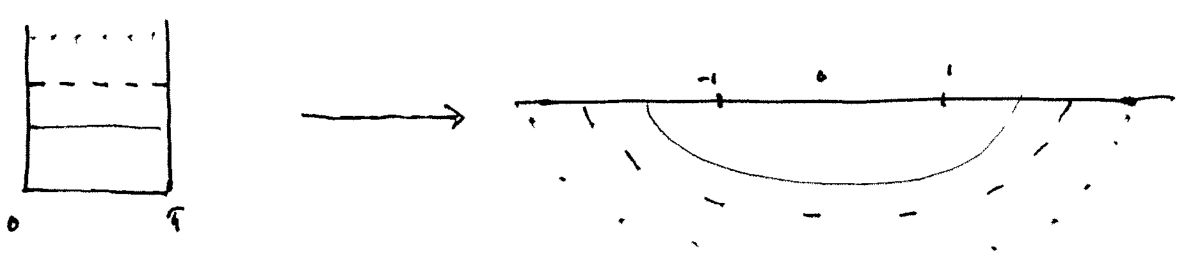
$HVS_s \xrightarrow{\sin z} UHP$

b) $\cos(-z) = \cos z$, $\cos \bar{z} = \overline{\cos z}$, $\cos(\pi + z) = -\cos \pi$, $\cos(z + 2\pi) = \cos z$

$(\cos z)' = -\sin z \Rightarrow (\cos z)' = 0$ for $z = k\pi$, $k \in \mathbb{Z}$

$\Rightarrow$ need only consider $HVS_c = \{z \mid \operatorname{Im} z > 0,\ 0 < \operatorname{Re} z < \pi\}$

$HVS_c \xrightarrow{\cos z} LHP$



5. Fixed Points

Dilation	$Tz = az,\ a \neq 1$	:	$0, \infty$
Translation	$Tz = z + a,\ a \neq 0$	:	∞
Inversion	$Tz = \frac{1}{z}$	:	$1, -1$

8 $\quad Tz = \dfrac{az+b}{cz+b}$ Möbius $\qquad T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ iff $\exists$ choice $a,b,c,d \in \mathbb{R}$

$\Leftarrow \quad T(0) = \frac{b}{d},\ T(1) = \dfrac{a+b}{c+d},\ T(-1) = \dfrac{-a+b}{-c+d}$ each belong to $\mathbb{R}_\infty$

hence $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$

$\Rightarrow \quad \exists\ x_2 \in \mathbb{R}_\infty \ni T(x_2) = 1,\quad x_3 \in \mathbb{R}_\infty \ni T(x_3) = 0,\quad x_4 \in \mathbb{R}_\infty \ni T(x_4) = \infty$

hence $Tz = (z, x_2, x_3, x_4) = \dfrac{x_2 - x_4}{x_2 - x_3}\ \dfrac{z - x_3}{z - x_4}$

$$= \frac{(x_2 - x_4) z + (x_2 - x_4)(-x_3)}{(x_2 - x_3) z + (x_2 - x_3)(-x_4)} = \frac{a z + b}{c z + d}$$

and by construction $a,b,c,d \in \mathbb{R}_\infty$

Note if any one x_2, x_3, x_4 is ∞ appropriately modify Tz

10 $\quad T: \mathbb{D} \to \mathbb{D} \quad \Rightarrow \quad \exists\ a \in \mathbb{D} \ni Ta = 0$

Case 1 $\quad a = 0 \quad \Rightarrow \quad T0 = 0\ \&\ T\infty = \infty \quad \Rightarrow Tz = \alpha z\ \&\ |\alpha| = 1$

Case 2 $\quad a \neq 0 \quad \Rightarrow \quad Ta = 0\ \&\ Ta^* = \infty \quad (a^* = \frac{1}{\bar{a}})$

$$\Rightarrow \quad Tz = \lambda \frac{z - a}{z - a^*} = -\lambda \bar{a} \frac{z - a}{1 - \bar{a} z}$$

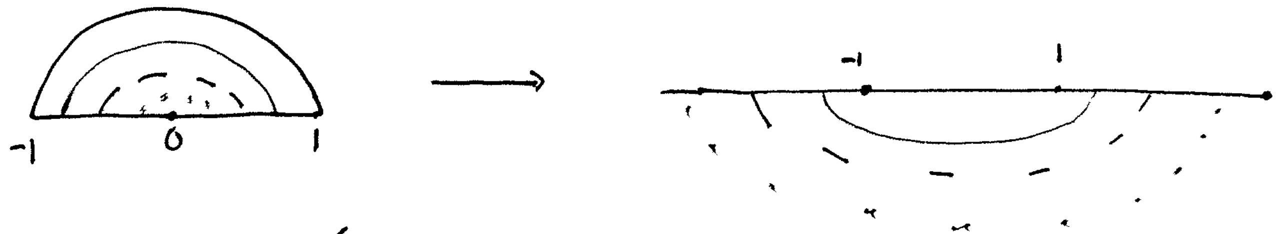
$$|T1| = 1 \quad \Rightarrow \quad |-\lambda \bar{a}| = 1$$

$$Tz = \alpha \frac{z - a}{1 - \bar{a} z} \quad \& \quad |\alpha| = 1$$

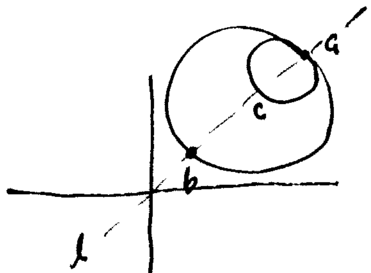
13. $f(z) = f\left(\frac{1}{z}\right)$, $f(\bar{z}) = \overline{f(z)}$, $f(-z) = -f(z)$

$\Rightarrow$ need only consider $\mathbb{D}_+ = \{ z \in \mathbb{D} \mid \operatorname{Im} z > 0 \}$

$$\mathbb{D}_+ \xrightarrow{f} LHP$$

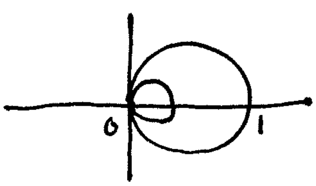


14.

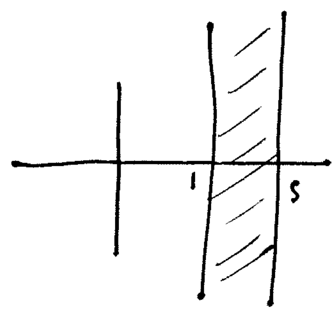


let l be line of symmetry and let b be the other intersection point of l with outer circle

$f_1 \downarrow \dfrac{z-a}{b-a}$



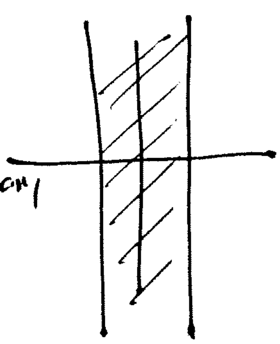
$\xrightarrow[\frac{1}{z}]{f_2}$



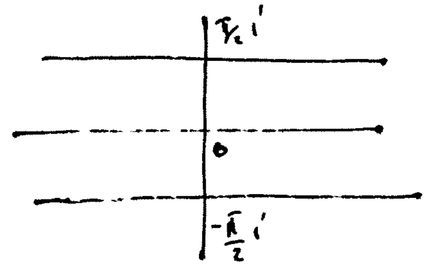
$s = f_2(f_1(c))$

f_3: $\left(z - \frac{s+1}{2}\right)$

symmetry about imaginary axis

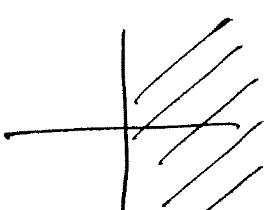


$\xrightarrow[i\alpha z]{f_4}$

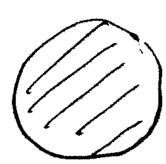


$$\alpha = \frac{\frac{\pi}{2}}{\frac{s-1}{2}}$$

f_5: e^z

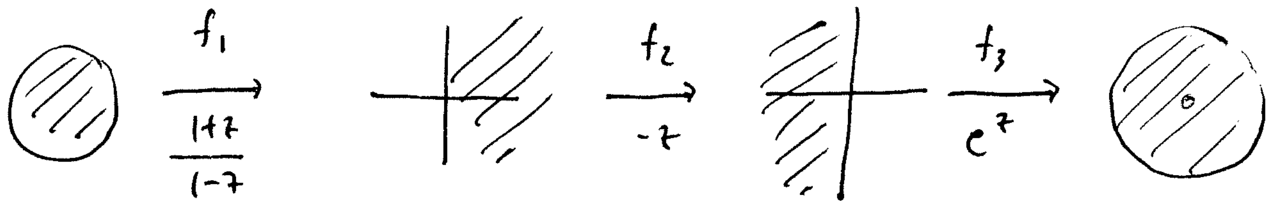


$\xrightarrow[\frac{z-1}{z+1}]{f_6}$



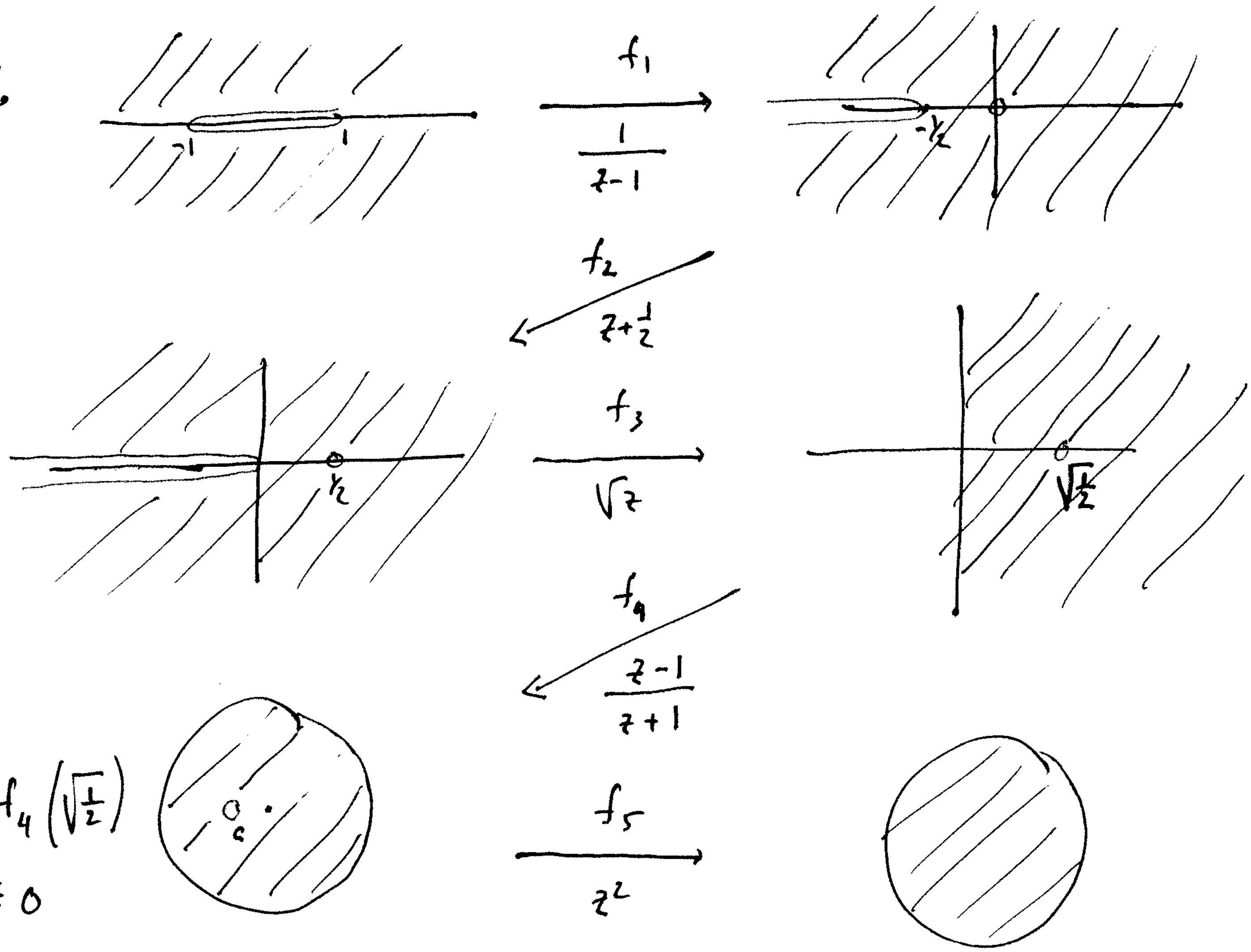
$f = f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$

15



$f = f_3 \circ f_2 \circ f_1$

16



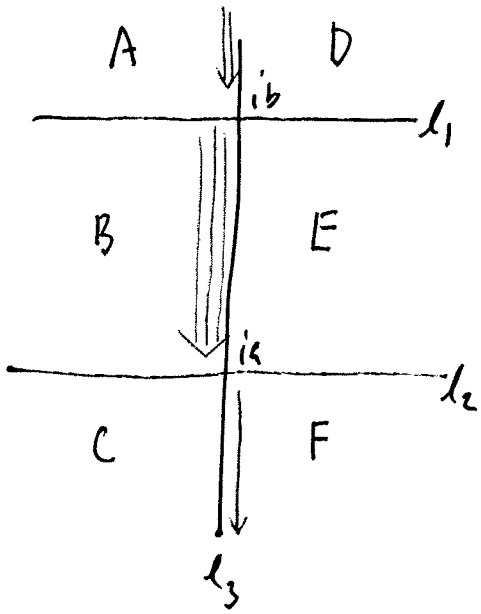
$a = f_4\left(\sqrt{\tfrac{1}{2}}\right)$

$a \neq 0$

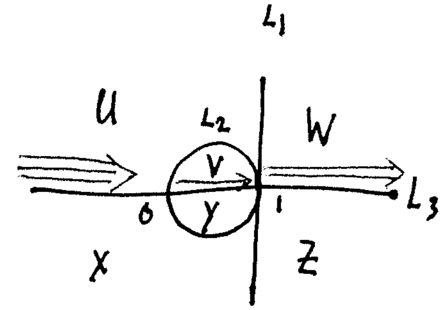
$f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$

17. Suppose $f \in \mathcal{H}(G)$, G region $\ni$ $f(G) \subset \mathcal{C}$ a circle.
Let c be center of $\mathcal{C}$ and $f_2(z) = z - c$. Let $g = f_2 \circ f$
Then $g(G)$ has constant modulus $\Rightarrow$ g constant (in class)
$\Rightarrow$ f constant.

18



$$\xrightarrow[\frac{z-ia}{z-ib}]{M}$$



$$M(l_j) = L_j$$

orient l_3 as

$$ia \longrightarrow \infty$$
$$\infty \Longrightarrow ib$$
$$ib \Rrightarrow ia$$

then L_3 is oriented as

$$0 \longrightarrow 1$$
$$1 \Longrightarrow \infty$$
$$\infty \Rrightarrow 0$$

$\therefore$ $M(F) = V$

$M(C) = Y$

$M(D) = W$

$M(A) = Z$

$M(E) = U$

$M(B) = X$

20 $\quad Sz = Tz \quad$ iff $\quad \exists \lambda \ni \begin{matrix} \alpha = \lambda a \\ \beta = \lambda b \\ \gamma = \lambda c \\ \delta = \lambda d \end{matrix}$

$\Leftarrow$ trivial

$\Rightarrow \quad Sz = \left(z, \frac{b-d}{a-c}, \frac{-b}{a}, \frac{-d}{c}\right)$

$Tz = \left(z, \frac{\beta-\delta}{\alpha-\gamma}, \frac{-\beta}{\alpha}, \frac{-\delta}{\gamma}\right)$

① $\frac{-b}{a} = \frac{-\beta}{\alpha} \Rightarrow \frac{\alpha}{a} = \frac{\beta}{b} \Rightarrow \exists x \ni \begin{matrix} \alpha = ax \\ \beta = bx \end{matrix}$

② $\frac{-d}{c} = \frac{-\delta}{\gamma} \Rightarrow \frac{\gamma}{c} = \frac{\delta}{d} \Rightarrow \exists y \ni \begin{matrix} \gamma = yc \\ \delta = yd \end{matrix}$

③ $\frac{b-d}{a-c} = \frac{\beta-\delta}{\alpha-\gamma} = \frac{bx-dy}{ax-cy}$

$(b-d)(ax-cy) = (bx-dy)(a-c) \quad \Rightarrow x = y$

22. $\quad T(z) = z \quad \Rightarrow \quad \overset{\text{exactly}}{z = 0, \infty} \quad$ iff $\quad T(z) = \alpha z \quad \alpha \neq 1$

$\Leftarrow$ trivial

$\Rightarrow \quad \begin{matrix} T0 = 0 & \Rightarrow & b = 0 \\ T\infty = \infty & \Rightarrow & c = 0 \end{matrix} \quad \Rightarrow \quad Tz = \frac{a}{d} z = \alpha z \quad \alpha \neq 1$

23. $\quad T0 = \infty, \quad T\infty = 0 \quad$ iff $\quad Tz = \frac{\alpha}{z}, \ \alpha \neq 0$

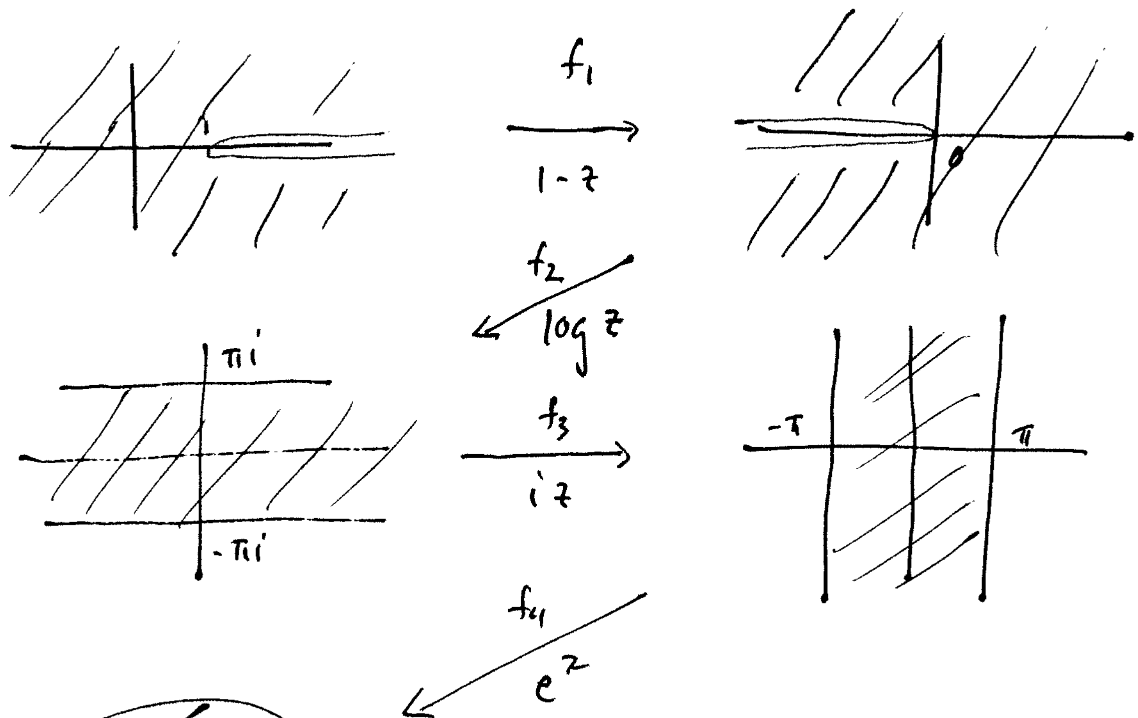
$\Leftarrow$ Trivial

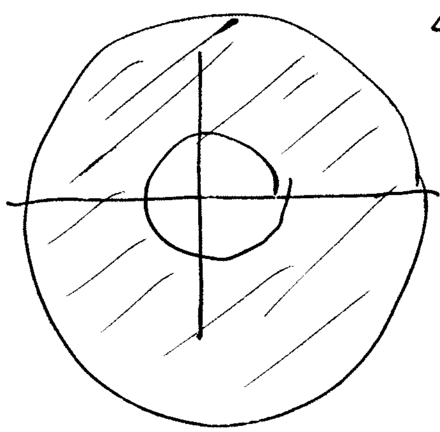
$\Rightarrow \quad \begin{matrix} T\infty = 0 & \Rightarrow & a = 0 \\ T0 = \infty & \Rightarrow & d = 0 \end{matrix} \qquad T(z) = \frac{b}{c}\frac{1}{z} = \frac{\alpha}{z}, \ \alpha \neq 0$

28 $\quad Tz = (1-z)^i = \exp^{i \log(1-z)}$ where log is principal branch of $\log z$

domain of T is $\mathbb{C} \setminus [1, \infty)$

$$Tz = e^{-\arg(1-z) + i \log|1-z|}$$





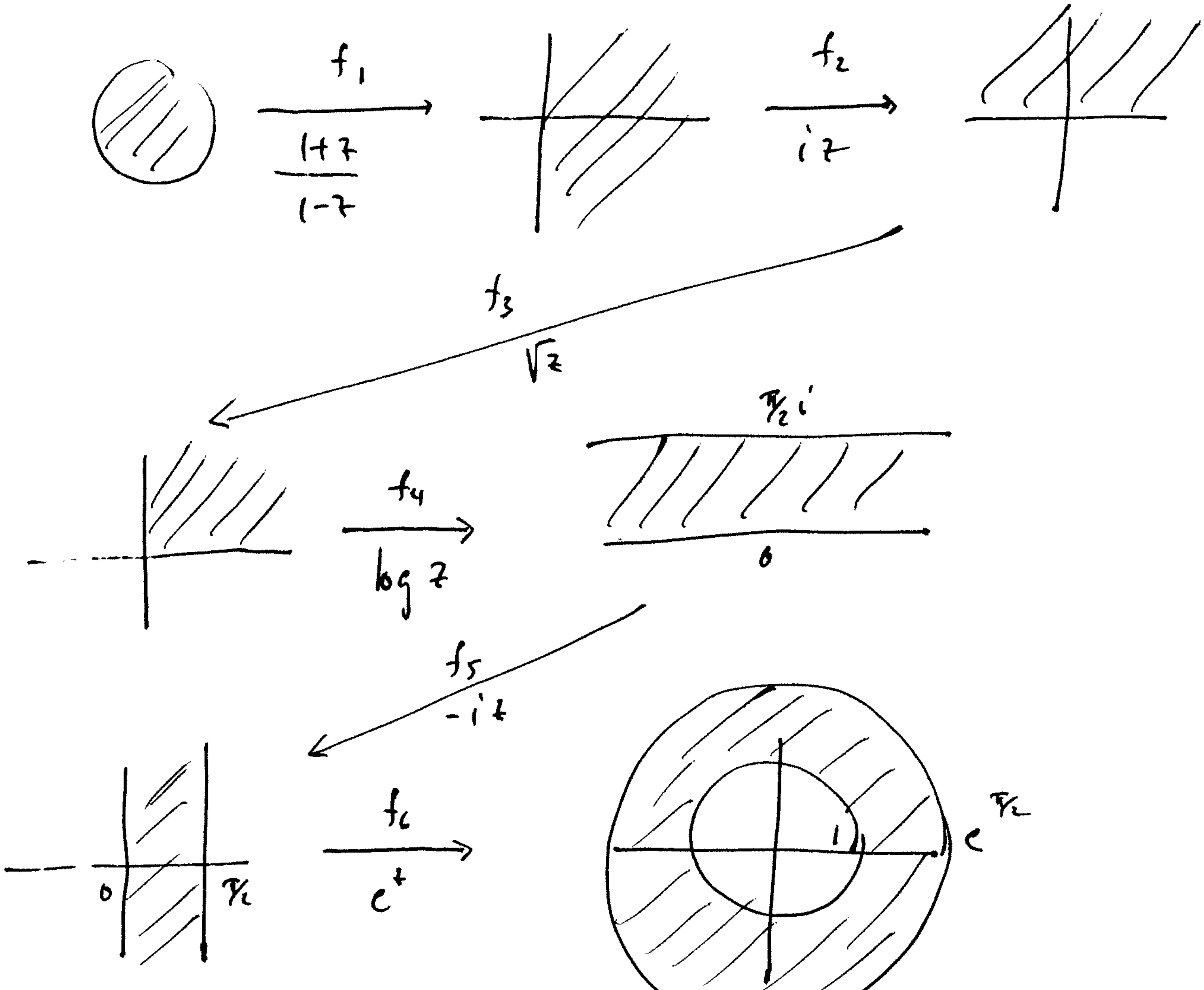
inner radius $e^{-\pi}$

outer radius e^{π}

$$T = f_4 \circ f_3 \circ f_2 \circ f_1$$

30.

a)



$$f(z) = f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1 (z)$$

b) Suppose $f\left(S(z)\right) = f(z)$ $\Rightarrow$

$$\exp\left\{-i \log\left[i \frac{1+S}{1-S}\right]^{1/2}\right\} = \exp\left\{-i \log\left[i \frac{1+z}{1-z}\right]^{1/2}\right\} \Rightarrow$$

$$-i \log\left[i\frac{1+S}{1-S}\right]^{1/2} = -i \log\left[i \frac{1+z}{1-z}\right]^{1/2} + 2\pi k i \quad \text{for some } k \in \mathbb{Z} \Rightarrow$$

$$\left[i \frac{1+S}{1-S}\right]^{1/2} = \left[i \frac{1+z}{1-z}\right]^{1/2} e^{-2\pi k} \Rightarrow \frac{1+S}{1-S} = \frac{1+z}{1-z} e^{-4\pi k} \Rightarrow$$

$$S = \frac{w-1}{w+1} \quad \text{where} \quad w = \frac{1+z}{1-z} e^{-4\pi k}$$

P.1. Given $\varepsilon > 0$ $\exists$ $N \ni n > N \Rightarrow e^{-\frac{1}{n}} < \frac{\varepsilon}{4}$

Let $\delta_1 = \frac{1}{N+1}$. Then for $z, w \in \overline{B(0,\delta_1)} \setminus \{0\}$

we have $|f(z) - f(w)| < \frac{\varepsilon}{2}$.

$\overline{B(0,1)} \setminus B(0,\delta_1)$ is compact and f is continuous on it.

By Thm 5.15 f is unif. cont on $\overline{B(0,1)} \setminus B(0,\delta_1)$; i.e.,

given $\varepsilon > 0$ $\exists$ $\delta_2 \ni |z-w| < \delta_2 \Rightarrow |f(z) - f(w)| < \frac{\varepsilon}{2}$

$\therefore$ On $B(0,1)\setminus\{0\}$ given $\varepsilon > 0$ choose $\delta = \min(\delta_1, \delta_2)$.

P.2. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ converges absolute $\forall z$

Case 1 $z > 0$, then $\frac{z^2}{2!} + \frac{z^3}{3!} + \cdots > 0 \Rightarrow e^z > 1+z$

Case 2 $z \le -1$, then $1+z \le 0 < e^z$

Case 3 $-1 < z < 0$ Let $x = -z$ so that $0 < x < 1$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots$$

but $\frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$ is (1) alternating (2) decreasing i.e., $\frac{x^n}{n!} > \frac{x^{n+1}}{(n+1)!}$ $\forall n$

$$\therefore \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots > 0$$

$$e^{-x} > 1 - x \iff e^z > 1+z$$

P. 3. a) $\cos 2z = 3i \qquad z = x+iy$

$\cos 2x \cosh 2y - i \sin 2x \sinh 2y = 3i \quad \Rightarrow$

$\cos 2x \cosh 2y = 0 \quad \& \quad -\sin 2x \sinh 2y = 3$

$\cosh s \neq 0$ for reals

$\therefore \cos 2x = 0$

$\Rightarrow 2x =$ odd mult $\frac{\pi}{2} = (2k+1)\frac{\pi}{2} \Rightarrow \sin 2x = \pm 1$

$2x = (4k+1)\frac{\pi}{2} \Rightarrow \sin 2x = +1 \Rightarrow \sinh 2y = -3$

$\Rightarrow y = \frac{1}{2}\sinh^{-1}(-3)$

$= -\frac{1}{2}\sinh^{-1}(3)$

$2x = (4k+3)\frac{\pi}{2} \Rightarrow \sin 2x = -1 \Rightarrow \sinh 2y = 3$

$\Rightarrow y = \frac{1}{2}\sinh^{-1} 3$

b) $\sin z = 8 \qquad z = x+iy$

$\sin x+iy = \sin x \cosh y + i \cos x \sinh y = 8 \Rightarrow$

$\sin x \cosh y = 8 \quad \& \quad \cos x \sinh y = 0$

If $\sinh y = 0 \Leftrightarrow y = 0$, then $\cosh y = 1$ and $\sin x = 8 \rightarrow \times$

$\therefore \cos x = 0 \Rightarrow x$ odd mult $\frac{\pi}{2}$ $x = (2k+1)\frac{\pi}{2}$

$\Rightarrow \sin x = \pm 1$

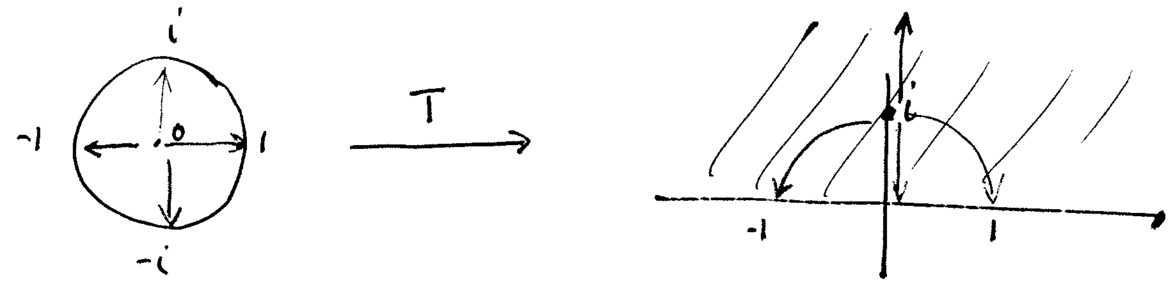
$x = (4k+1)\frac{\pi}{2} \Rightarrow \sin x = 1 \Rightarrow \cosh y = 8$

$\Rightarrow y = \pm \cosh^{-1} 8$

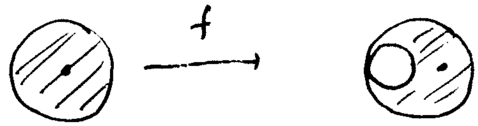
$x = (4k+3)\frac{\pi}{2} \Rightarrow \sin x = -1 \Rightarrow \cosh y = -8 \quad \times$

P. 4

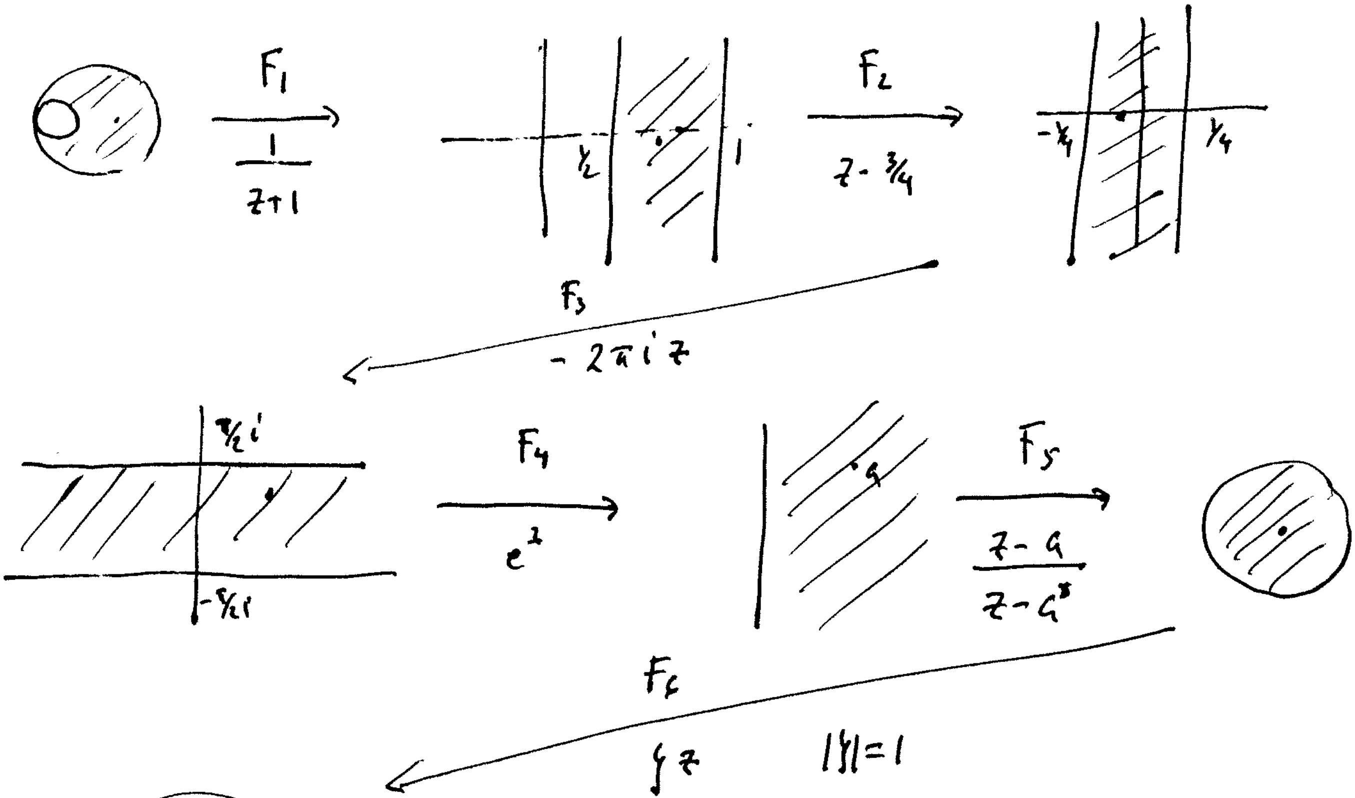
$$Tz = \frac{i+1}{i-1}\,\frac{z-1}{z+1} = -i\,\frac{z-1}{z+1} = i\,\frac{1-z}{1+z}$$

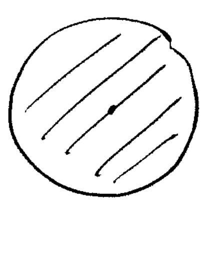


P. 5



$f = F^{-1}$





$F_1\left(\frac{1}{2}\right) = \frac{2}{3} \quad F_2\left(\frac{2}{3}\right) = -\frac{1}{12} \quad F_3\left(-\frac{1}{12}\right) = \frac{\pi}{6} i$

$F_4\left(\frac{\pi}{6} i\right) = e^{\frac{\pi}{6} i} = a \qquad F_5(a) = 0 \qquad F_6(0) = 0$

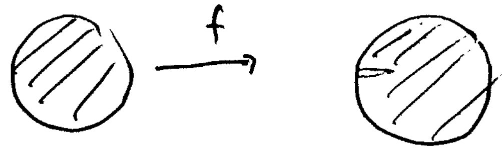
$$F = F_6 \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1$$

choose $\gamma = e^{i\theta}$ so that

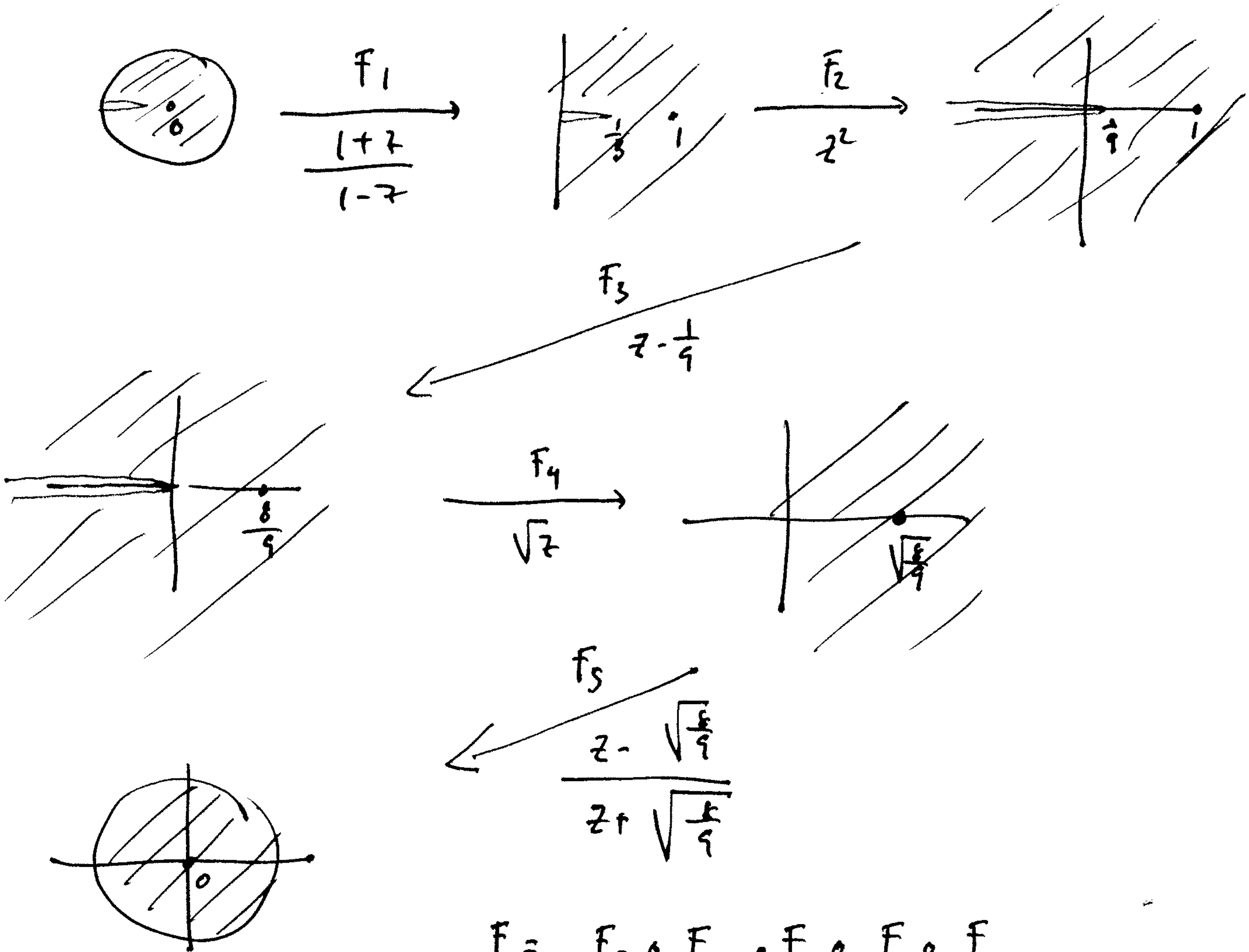
$$F'\left(\tfrac{1}{2}\right) = F_6'(0)\, F_5'\left(e^{\frac{\pi}{6} i}\right) F_4'\left(\tfrac{\pi}{6} i\right) F_3'\left(-\tfrac{1}{12}\right) F_2'\left(\tfrac{2}{3}\right) F_1'\left(\tfrac{1}{2}\right)$$

has argument 0

P. 6.



$f = F^{-1}$



$F = F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1$