Review Exam III
Complex Analysis

Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam
Underlined Homework Exercises: Problems may be asked for on exam
Double Underlined Homework Exercises: Similar problems will be asked for on exam
Double Underlined Named Theorems/Results: Statements may be asked for on exam

Chapter 7.1

Homework 7.1 Page 150 1, 2, 4, 5, 7, 8

Definition. Let G be a region and let (Ω, d) be a complete metric space. Then, $C(G,\Omega) = \ldots$

Proposition. Let G be a region. Then there exists a sequence of subsets $\{K_n\}$ of G such that

(i) $K_n \subset\subset G$

(ii) $K_n \subset \text{int}(K_{n+1})$

(iii) $\bigcup_{n=1}^{\infty} K_n = G$

(iv) $K \subset\subset G$ implies $K \subset K_n$ for some $n \in \mathbb{N}$

Lemma If (S,d) is a metric space, then (S,μ) is a metric space, where $\mu(s,t) = \dfrac{d(s,t)}{1+d(s,t)}$. A set O is open in (S,d) if and only if O is open in (S,μ) .

Definition. For $K \subset\subset G$ and $f,g \in C(G,\Omega)$, let $\rho_K(f,g) = \sup_{z\in K} d(f(z),g(z))$,

$$\sigma_K(f,g) = \frac{\rho_K(f,g)}{1+\rho_K(f,g)},\ B_{\rho_K}(f,\delta) = \{g : \rho_K(f,g) < \delta\}.$$

Definition. For $\{K_n\}$ a compact exhaustion of a region G and for $f,g \in C(G,\Omega)$ let $\rho(f,g) = \ldots$

Proposition. $(C(G,\Omega),\rho)$ is a metric space.

Lemma 1.7 (i) Given $\varepsilon > 0$ there exists $\delta > 0$ and $K \subset\subset G$ such that for $f,g \in C(G,\Omega)$

$$\rho_K(f,g) < \delta \ \Rightarrow\ \rho(f,g) < \varepsilon$$

(ii) Given $\delta > 0$ and there exists $\varepsilon > 0$ such that

$$\rho(f,g) < \varepsilon \ \Rightarrow\ \rho_K(f,g) < \delta$$

Lemma 1.10 (i) A set $O \subset \mathcal{C}(G,\Omega)$ is open if and only if for each $f \in O$ there exists $\delta > 0$ and $K \subset\subset G$ such that $O \supset B_{\rho_K}(f,\delta)$

(ii) A sequence $\{f_n\} \subset \mathcal{C}(G,\Omega)$ converges to f (in the ρ metric) if and only if for each $K \subset\subset G$ $\{f_n\}$ converges to f in the ρ_K metric.

Proposition $(\mathcal{C}(G,\Omega),\rho)$ is a complete metric space.

Definition. A set $F \subset \mathcal{C}(G,\Omega)$ is normal . . .

Proposition. A set $F \subset \mathcal{C}(G,\Omega)$ is normal if and only if $\overline{F}$ is compact.

Proposition. A set $F \subset \mathcal{C}(G,\Omega)$ is normal if and only if for each $\delta > 0$ and $K \subset\subset G$ there exist functions $f_1, f_2, \cdots, f_n \in F$ such that $F \subset \bigcup_{k=1}^{n} B_{\rho_K}(f_k,\delta)$.

Definition. A set $F \subset \mathcal{C}(G,\Omega)$ is equicontinuous at a point $z_0 \in G$ if . . .

Definition. A set $F \subset \mathcal{C}(G,\Omega)$ is equicontinuous on a set $E \subset G$ if . . .

Proposition. Suppose a set $F \subset \mathcal{C}(G,\Omega)$ is equicontinuous at each point of G. Then, F is equicontinuous on each $K \subset\subset G$.

Arzela-Ascoli Theorem

Chapter 7.2

Homework 7.2 Page 154 4, 6, 8, 10, 13

Definition Let G be a region. $\mathcal{A}(G) = \cdots$

Theorem Let G be a region. Let $\{f_n\} \subset \mathcal{A}(G)$ and let $f \in \mathcal{C}(G,\mathbb{C})$. If $f_n \to f$, then $f \in \mathcal{A}(G)$ and $f_n^{(k)} \to f^{(k)}$ for each $k \geq 1$.

Hurwitz's Theorem

Corollary Let G be a region. Let $\{f_n\} \subset \mathcal{A}(G)$ and $f \in \mathcal{A}(G)$ be such that $f_n \to f$. If each f_n is non-vanishing on G, then either f is non-vanishing on G or else $f \equiv 0$.

Definition A set $\mathcal{F} \subset \mathcal{A}(G)$ is locally bounded if . . .

Lemma A set $\mathcal{F} \subset \mathcal{A}(G)$ is locally bounded if and only if for each $K \subset\subset G$ there exists a constant M such that $|f(z)| \le M$ for all $f \in \mathcal{F}$ and for all $z \in K$.

Montel's Theorem

Chapter 7.4

Homework 7.4 Page 163 4, 5, 6, 7

Definition A region G_1 is conformally equivalent to a region G_2 if . . .

Riemann Mapping Theorem

Chapter 7.5

Homework 7.5 Page 173 4, 5, 6, 7, 9

Definition Let $\{z_n\} \subset \mathbb{C}$. Then, the infinite product $\prod_{n=1}^{\infty} z_n = \cdots$

Proposition Let $\operatorname{Re} z_n > 0$ for all n. Then, the product $\prod_{n=1}^{\infty} z_n$ converges to a non-zero number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proposition Let $\operatorname{Re} z_n > 0$ for all n. Then, the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n - 1$ converges absolutely.

Definition Let $\operatorname{Re} z_n > 0$ for all n. The product $\prod_{n=1}^{\infty} z_n$ converges absolutely if . . .

Corollary Let $\operatorname{Re} z_n > 0$ for all n. Then, the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if series $\sum_{n=1}^{\infty} z_n - 1$ converges absolutely.

Theorem Let G be a region. Let $\{f_n\} \subset \mathcal{A}(G)$ be such that no f_n is identically 0. If $\sum [f_n(z)-1]$ converges in $\mathcal{A}(G)$, then $\prod f_n(z)$ converges in $\mathcal{A}(G)$. Further, each zero of $\prod f_n(z)$ is a zero of one or more of the factors $f_n(z)$.

<u>Definition</u> An elementary factor $E_p(z) = \cdots$

Lemma If $|z| \le 1$, then $|E_p(z)-1| \le |z|^{p+1}$

Theorem Let $\{a_n\} \subset \mathbb{C}$ be such that $\lim_{n\to\infty} |a_n| = \infty$, $a_n \ne 0$ for all *n*. If $\{p_n\}$ is a sequence of integers such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty \qquad (*)$$

for all $r > 0$, then $\prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ converges to an entire function whose zero set is precisely $\{a_n\}$.

Furthermore, (*) is always satisfied if $p_n = n-1$.

<u>Weierstrass Factorization Theorem</u>

Chapter 7.6

Theorem. $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)$.

Chapter 7.7

Homework   7.7 Page 185 1, 2, 3, 7, 8

<u>Definition</u> The gamma function $\Gamma(z) = \cdots$

<u>Gauss's Formula</u>

Gauss's Functional Equation For $z \ne 0, -1, -2, \cdots$, $\Gamma(z+1) = z\Gamma(z)$.

<u>Bohr-Mollerup Theorem</u>

Integral Representation For $\operatorname{Re} z > 0$, $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$.

Lemma $\left\{ \left(1 + \frac{z}{n} \right)^n \right\}$ converges to e^z in $\mathcal{A}(G)$

Chapter 7.8

<u>Definition</u> The Riemann zeta function $\zeta(z) = \cdots$

Integral Representation 1. For $\operatorname{Re} z > 1$, $\zeta(z)\Gamma(z) = \int_0^\infty \frac{1}{e^t - 1} t^{z-1}\, dt$.

Extension 1. For $\operatorname{Re} z > 0$, $\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt + \frac{1}{z-1} + \int_1^\infty \frac{1}{e^t - 1} t^{z-1}\, dt$

Integral Representation 2. For $0 < \operatorname{Re} z < 1$, $\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$.

Extension 2. For $-1 < \operatorname{Re} z < 1$,

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$$

Integral Representation 3. For $-1 < \operatorname{Re} z < 0$, $\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$

Riemann's Functional Equation

Theorem $\zeta(z) \in \mathcal{A} \quad (\mathbb{C} \setminus \{1\})$ with a simple pole at $z = 1$ with residue 1. Outside of the strip $0 \le \operatorname{Re} z \le 1$, $\zeta(z)$ is non-vanishing except for simple zeros at $z = -2, -4, -6, \cdots$.

Riemann Hypothesis

Euler's Theorem For $\operatorname{Re} z > 0$, $\zeta(z) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - p_n^{-z}} \right)$, where $\{p_n\}$ is an enumeration of the prime numbers.