Lemma. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial of degree *n*. Then, for each ε, $0 < \varepsilon < 1$, there exists an $R > 0$ such that for $|z| > R$

$$(1-\varepsilon)|a_n||z|^n < |p(z)| < (1+\varepsilon)|a_n||z|^n$$

Let *r* be rational function, $r : \mathbb{R}_\infty \to \mathbb{R}_\infty$. For $r = \dfrac{p}{q}$, where *p* and *q* have no common zeros, define deg(*r*) = deg(*p*) - deg(*q*) and let pole set of *r*, P_r, be given as $P_r = Z_q$.

Theorem. Let *r* be rational function, $r : \mathbb{R}_\infty \to \mathbb{R}_\infty$. Suppose

(1) $\deg(r) \le -2$ and

(2) $P_r \cap \mathbb{R} = \varnothing$.

Then,

$$\int_{-\infty}^{\infty} r(x)dx = 2\pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \operatorname{Res}(r(z); a_k)$$

Corollary. Suppose additionally, *r* is even. Then,

$$\int_{0}^{\infty} r(x)dx = \pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \operatorname{Res}(r(z); a_k)$$

Examples. $\displaystyle\int_{-\infty}^{\infty} \frac{x+1}{x^4+1}dx$, $\displaystyle\int_{0}^{\infty} \frac{x^2+1}{x^4+1}dx$,

Exceptions. $\displaystyle\int_{0}^{\infty} \frac{x}{x^4+1}dx$, $\displaystyle\int_{0}^{\infty} \frac{1}{x^3+1}dx$

Theorem. Let r be rational function, $r:\mathbb{R}_\infty \to \mathbb{R}_\infty$. Suppose

(1) $\deg(r) \le -1$ and

(2) $P_r \cap \mathbb{R} = \varnothing$.

Then,

$$\int_{-\infty}^{\infty} \cos(x) r(x) dx = \mathrm{Re}\left(2\pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \mathrm{Res}(e^{iz} r(z); a_k) \right) \text{ and}$$

$$\int_{-\infty}^{\infty} \sin(x) r(x) dx = \mathrm{Im}\left(2\pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \mathrm{Res}(e^{iz} r(z); a_k) \right)$$

Corollary. Suppose additionally, r is even. Then,

$$\int_{0}^{\infty} \cos(x) r(x) dx = \mathrm{Re}\left(\pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \mathrm{Res}(e^{iz} r(z); a_k) \right)$$

Alternately, suppose additionally, r is odd. Then,

$$\int_{0}^{\infty} \sin(x) r(x) dx = \mathrm{Im}\left(\pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \mathrm{Res}(e^{iz} r(z); a_k) \right)$$

Examples. $\displaystyle\int_0^\infty \frac{x\sin(x)}{x^2+a^2} dx, \;\; a \in \mathbb{R} \setminus \{0\}$

Theorem. Let r be rational function, $r:\mathbb{R}_\infty \to \mathbb{R}_\infty$. Suppose

(1) $\deg(r) \le -1$ and

(2) $P_r \cap \mathbb{R} = \{0\}$

(3) r has simple pole at 0

Then,

$$P.V.\int_{-\infty}^{\infty} \sin(x) r(x) dx = \operatorname{Im}\left(2p i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \operatorname{Res}(e^{iz} r(z); a_k) + p i \operatorname{Res}(e^{iz} r(z); 0) \right)$$

Corollary. Suppose additionally r is odd. Then,

$$\int_{0}^{\infty} \sin(x) r(x) dx = \operatorname{Im}\left(p i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \operatorname{Res}(e^{iz} r(z); a_k) + \frac{p}{2} i \operatorname{Res}(e^{iz} r(z); 0) \right)$$

Example. $\int_{0}^{\infty} \frac{\sin(x)}{x} dx$

Extended Example. $\int_{0}^{\infty} \frac{\sin^2(x)}{x^2} dx$

Theorem. Let r be rational function, $r : \mathbb{R}_\infty \to \mathbb{R}_\infty$. Suppose

(1) $\deg(r) \le -2$ and

(2) $P_r \cap \{\mathbb{R}^+ \cup \{0\}\} = \varnothing$.

Suppose $0 < a < 1$. Then,

$$(1 - e^{2p i a}) \int_{0}^{\infty} x^a r(x) dx = 2p i \sum_{a_k \in P_r} \operatorname{Res}(z^a r(z), a_k)$$

where $z^a = \exp(a \log_0 z)$ and $\log_0 z = \log|z| + i \arg_0 z$, $0 < \arg_0 z < 2p$.

Example. $\int_{0}^{\infty} \frac{x^a}{x^2 + 3x + 2} dx$, $0 < a < 1$

Extended Example. $\int_{0}^{\infty} \frac{\log x}{(1 + x^2)^2} dx$

Theorem. Let $R(x, y)$ be rational in x, y. Suppose $R(\cos(q), \sin(q))$ is continuous on $[0, 2p]$. Then,

$$\int_{0}^{2p} R(\cos(q), \sin(q))\, dq = \int_{|z|=1} R\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \frac{dz}{iz}$$

Example. $\int_{0}^{2p} \frac{dq}{2+\cos q}$