# Key Exam II

1a) $f(z)$ is even. Therefore $a_{2k+1} = 0 \quad \forall k \in \mathbb{Z}$. Specifially $a_{-1} = 0$

b) $$f(z) = \left(1 + 2z - z^2 + 2z^3 - z^4 + 2z^5 + z^6\right)\left(1 - \frac{2}{z^2} + \frac{1}{2}\frac{4}{z^4} - \frac{1}{6}\frac{8}{z^6} + \cdots\right)$$

$$= z^6 + 2z^5 - 3z^4 - 2z^3 + 3z^2 + 2z + \frac{7}{3} - \frac{\frac{8}{3}}{z} + \cdots$$

$$a_{-1} = -\frac{8}{3}$$

2a) by Thm Pg 2 on Int. Topics $\quad I = \operatorname{Re}\left\{\pi i \left(\operatorname{Res}(f,i) + \operatorname{Res}(f,2i)\right)\right\}$

where $$f(z) = \frac{e^{i2z}}{(z^2+1)(z^2+4)}$$

$$\operatorname{Res}(f,i) = \left.\frac{e^{i2z}/(z^2+4)}{2z}\right|_{z=i} \qquad \operatorname{Res}(f,2i) = \left.\frac{e^{i2z}/(z^2+1)}{2z}\right|_{z=2i}$$

$$\therefore \quad I = \pi\left(\frac{e^{-2}}{6} - \frac{e^{-4}}{12}\right)$$

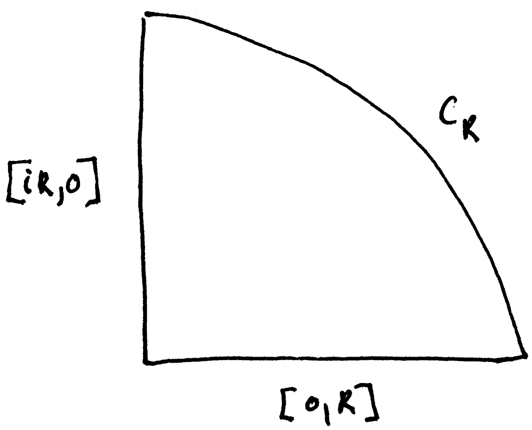
b) by Thm Pg 3 on Int. Topics $\quad I = \operatorname{Im}\left\{\pi i \operatorname{Res}(f,i) + \frac{\pi}{2} i \operatorname{Res}(f,0)\right\}$

where $$f(z) = \frac{e^{i2z}}{z(z^2+1)^2}$$

$$\operatorname{Res}(f,i) = \left.\left(\frac{e^{i2z}}{z(z+i)^2}\right)'\right|_{z=i} \qquad \operatorname{Res}(f,0) = \left.\frac{\frac{e^{i2z}}{(z^2+1)^2}}{1}\right|_{z=0}$$

$$\therefore \quad I = \pi\left(\frac{1}{2} - e^{-2}\right)$$

3. Let $f(z) = z^4 + z^3 - 2z^2 + 2z + 4$. Consider contour




On $[0,R]$ $z = x + i0$

on $[0,1]$ $f(x) \geq x^4 + x^3 + 4 \geq 4$

on $[1,R]$ $f(x) \geq 2x + 4 \geq 4$ } $\arg f(z) = c$

On $C_R$ $\Delta \arg f(z) \approx 2\pi$

On $[iR, 0]$ $z = 0 + iy$

$f(iy) = y^4 + 2y^2 + 4 + i(2y - y^3)$

$\Rightarrow f([iR, 0])$ lies in RHP for all $z = iy$

$\therefore \Delta_{[0,R] \cup C_R \cup [iR, 0]} \arg f(z) = 2\pi$

$\Rightarrow f$ has 1 root in $Q_I$.

4. Consider $b_R(z) = \dfrac{z}{2R - z}$. $b : \{\operatorname{Re} z < R\} \to |z| < 1$

Consider $g(z) = b_A \circ f(rz) : \mathbb{D} \to \mathbb{D}$, $g(0) = 0$

by Schwarz's Lemma $|g(z)| \leq |z|$

or $$\left| \frac{f(rz)}{2A - f(rz)} \right| \leq |z|$$

$$\Rightarrow \quad |f(rz)| \leq \frac{|z|\, 2A}{1 - |z|} \quad \text{for } z \in \mathbb{D}$$

or

$$|f(z)| \leq \frac{2A\,|z|}{r - |z|} \quad \text{for } z \in B(0,r)$$

5. Let $f \in \mathcal{F}$. Then $f < 2i\frac{1+z}{1-z}$. Hence $|f'(0)| \leq 4$. Since $2i\frac{1+z}{1-z} \in \mathcal{F}$, $\alpha = 4$

6. Since $u \in Harm(\mathbb{C})$ and $\mathbb{C}$ is simply connected $\exists$ $v \in Harm(\mathbb{C})$ $\ni$ $f = u+iv \in \mathcal{H}(\mathbb{C})$. Let $g = e^{-f}$. Then $g$ is entire and $|g(z)| \leq e^{-u} \leq e^{2}$. By Liouville's Thm $g$ is constant. Since $f$ cont. $\Rightarrow$ $f$ const. $\Rightarrow$ $u$ constant

7a) If $u$ and $v$ are harmonic conjugates, then $f$ $u+iv$ is analytic hence $g = f^2$ is analytic. $\underline{Im}\, g$ is therefore harmonic, $\underline{Im}\, g = 2uv \Rightarrow uv$ is harmonic

b) $u(x,y) = x$ is harmonic $v(x,y) = x$ is harmonic.
$u \cdot v = x^2$ is not harmonic

8. Let $A = \{z \in G \mid u(z) = u(a)\}$. $A$ is closed because $u$ is cont. on $G$. Show $A$ is open using same proof as given in Conway p. 253-254. Since $G$ connected, $A \neq \emptyset$, then $A = G$ and $u$ is constant.

9. Let $g_n = f - f_n$. It suffices to show for any $K \subset\subset G$ that $g_n \to 0$ uniformly on $K$. Let $\varepsilon > 0$. Let $K_n = \{z \in K : g_n(z) \geq \varepsilon\}$. We have $K_n \supset K_{n+1}$ from the monotonicity of $\{g_n\}$. Each $K_n$ is compact (since each $K_n$ is a closed subset of $K$). Since for each $z \in K$ $\lim_{n\to\infty} g_n(z) = 0$, we have $\bigcap_{n=1}^{\infty} K_n = \phi$. The compactness of $K$ implies $\exists\, N \ni K_N = \phi$. Hence, $\{g_n\} \to 0$ uniformly on $K$.