Key Exam II

1. a) $\sum n^2 (2z-i)^n = \sum n^2 2^n \left(z-\frac{i}{2}\right)^n$

$\frac{1}{R} = \lim_{n\to\infty} |n^2 2^n|^{\frac{1}{n}} = 2 \Rightarrow R = \frac{1}{2}$

Note $\limsup |n^2 2^n|^{\frac{1}{n}} = \lim_{n\to\infty} |n^2 2^n|^{\frac{1}{n}}$

b) $\frac{1}{R} = \limsup \left(\frac{1}{|(3+i^n)^n|}\right)^{\frac{1}{n}} = \limsup \frac{1}{|3+i^n|}$

$\left\{\frac{1}{|3+i^n|}\right\}_{n=1}^{\infty} = \left\{\frac{1}{|3+i|}, \frac{1}{2}, \frac{1}{|3-i|}, \frac{1}{4}, \dots\right\}$

$\therefore \limsup \frac{1}{|3+i^n|} = \frac{1}{2} \Rightarrow R = 2$

2. Let $f = u+iv$. C-R eq. $\Rightarrow$ $u_x = v_y$ ① $\quad v_x = -u_y$ ②

$\bar{f} \in \mathcal{A}(G)$ & C-R eq $\Rightarrow$ $u_x = -v_y$ ③ $\quad v_x = u_y$ ④

① & ③ $\Rightarrow$ $u_x = 0$

② & ④ $\Rightarrow$ $v_x = 0$

Since $f'(z) = u_x(x,y) + i\, v_x(x,y) = 0$ then Prop 2.10 $\Rightarrow$ $f$ constant

3. a) let $f(z) = \cosh^2 z - \sinh^2 z$

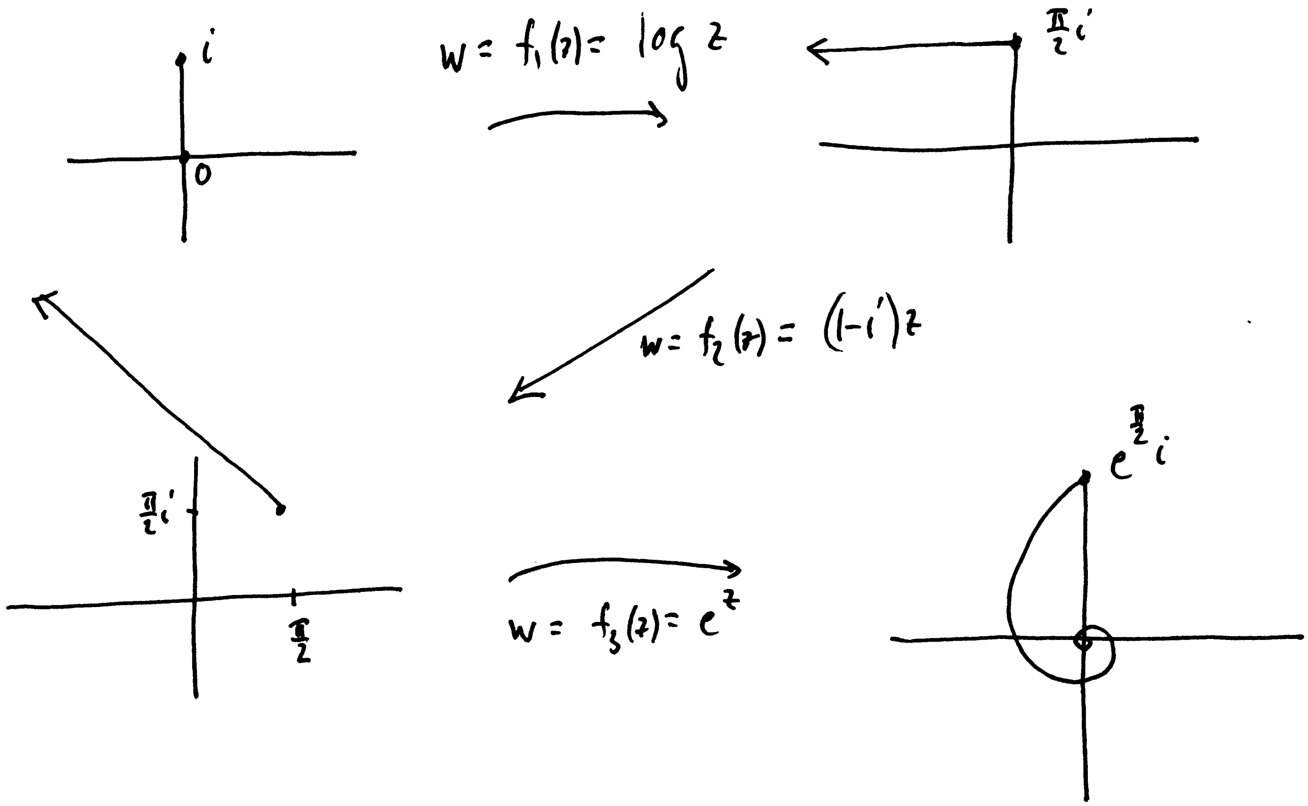
then $f'(z) \equiv 0$. Hence Prop 2.10 $\Rightarrow$ $f(z) = f(0) = 1$

b) $\cos^2 z - \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 =$

$\frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz} + e^{2iz} - 2 + e^{-2iz}\right) = \frac{1}{2}\left(e^{2iz}+e^{-2iz}\right) = \cos 2z$

4. $f(z) = z^{1-i} = e^{(1-i)\log z} = e^{(1-i)(\log|z| + i\arg z)}$

$= e^{\log|z| + \arg z + i(\arg z - \log|z|)}$




5. a) Solve $(z, 1+i, 0, 1-i) = \left(w, \frac{i}{i-1}, \frac{1}{2}, \frac{i}{i+1}\right)$ for $w = Mz$

or

Let $Mz = \dfrac{az+b}{cz+d} = \dfrac{\tilde{a}z+1}{\tilde{c}z+\tilde{d}}$ and

solve

$$\begin{cases} \dfrac{\tilde{a}(1+i)+1}{\tilde{c}(1+i)+\tilde{d}} = \dfrac{i}{i-1} = \dfrac{1-i}{2} \\[2ex] \dfrac{\tilde{a}(0)+1}{\tilde{c}(0)+\tilde{d}} = \dfrac{1}{2} \quad \Rightarrow \tilde{d} = 2 \\[2ex] \dfrac{\tilde{a}(1-i)+1}{\tilde{c}(1-i)+\tilde{d}} = \dfrac{i}{i+1} = \dfrac{1+i}{2} \end{cases}$$

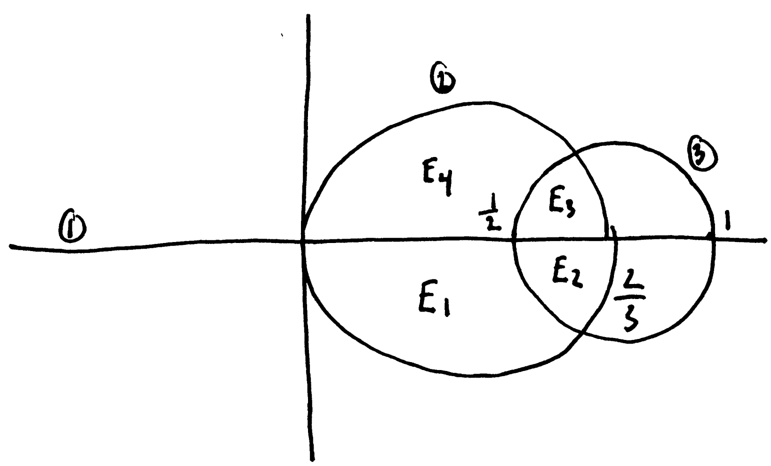
$\tilde{a} = \tilde{c} = -1 \quad \Rightarrow \quad Mz = \dfrac{-z+1}{-z+2} = \dfrac{z-1}{z-2}$

5 b) Since pole of $M = 2$ and $M$ has real coef.

① $M$ maps $\mathbb{R} \to \mathbb{R}$

② $M$ maps $|z|=1$ to circle which intersects $\mathbb{R}$ orthogonally at $M(-1) = \frac{2}{3}$ and $M(1) = 0$

③ $M$ maps imaginary axis to circle which intersects $\mathbb{R}$ orthogonally at $M(0) = \frac{1}{2}$ and $M(\infty) = 1$




$E_j = M(D_j)$