Instructions: Solve 13 of the problems 1–15. If you solve more than 13 problems, you must clearly mark which 13 you want to have graded. For full credit, you must show complete, correct, legible work. Read carefully before you start working. No books or notes are allowed. Calculators are allowed, phones and PDAs are not.

1. Construct a truth table for the given compound statement.

\[(p \rightarrow q) \lor (\sim p \land \sim q)\]

2. James has set up an ordinary annuity to save for his retirement. If he wants to have $70,000 in his account after 15 years and the annuity has an annual interest rate of 7.5% compounded monthly, how much should his monthly payments be?

3. Consider the following graph:

(a) Can the graph be traced? Explain your answer using Euler’s theorem.
(b) If the given graph is Eulerian, find an Euler circuit in it. If it is not Eulerian, first Eulerize it and then find an Euler circuit. Begin the Euler circuit at vertex A.

4. An apartment complex has three buildings. Building A has 120 units, building B has 141 units, and building C has 39 units. An 11-person committee will set rules to govern the complex.

(a) Apportion this committee based on the number of units per building using Hamilton’s method.
(b) Now increase the number of members in the committee by one and reapportion the committee (again using Hamilton’s method).
(c) Does an apportionment paradox occur? Explain your answer.

5. If you randomly select a single card from a standard 52-card deck, what is the probability that you draw a heart or a face card?

6. Find the mean, median, and mode of the following distribution.

12, 11, 7, 9, 8, 6, 6, 4, 5, 10, 1, 2
7. Consider the following voting preference table:

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>5</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

(a) Determine who wins the election according to the plurality method.
(b) Determine who wins the election according to the Borda Count method.
(c) Using the Borda count method, was the Majority Criterion violated? Explain your answer.

8. Alex, Becca, and Carly play on the same soccer team. Last season, they scored a total of 35 goals. If Becca scored three fewer goals than Alex, and Carly scored two more goals than Alex, how many goals did each soccer player score?

9. Use the weighted graph given below to answer the following questions:

(a) How many Hamilton circuits would you have to consider if you planned to use the brute force algorithm to find a path that minimizes the traveling cost?
(b) Use the nearest neighbor algorithm to find a Hamilton circuit beginning at vertex A.
(c) What is the weight of the path found in part b?

10. Determine whether the following syllogism is valid or invalid using Euler diagrams.

   Some animals are dangerous.
   All lions are animals.

   Some lions are dangerous.
11. Use the unpaid balance method to find the finance charge on the credit card account. Last month’s balance, the payment, the annual interest rate, and any other transactions are stated below.

- Last month’s balance: $509
- Payment: $208
- Annual interest rate: 19%
- Bought shoes: $89
- Bought jacket: $127

12. Determine which state is more poorly represented: State A has a population of 488,895 people and 11 representatives; State B has a population of 325,098 people and 9 representatives.

13. Consider the weighted voting system

\[ [11 : 3, 4, 5, 10] \]

where the weights represent voters A, B, C, and D respectively.

(a) List all coalitions, state their weights, and identify the winning coalitions. For each winning coalition, determine the critical voters.

(b) Compute the Banzhaf power index for each voter.

14. You are playing a game in which a single die is rolled. If you roll a 2 or a 5, you win $36, otherwise you lose $36. Is the game fair? Explain your answer.

15. Consider a normal distribution with a mean of 23 and a standard deviation of 3.

(a) What z-score corresponds to the raw score 26?

(b) Use the 68-95-99.7 Rule to determine what percentage of values would be below 26.
THE COMPOUND INTEREST FORMULA Assume that an account with principal $P$ is paying an annual interest rate $r$ and compounding is being done $m$ times per year. If the money remains in the account for $n$ time periods, then the future value, $A$, of the account is given by the formula

$$A = P\left(1 + \frac{r}{m}\right)^n.$$ 

Notice that in this formula, we have replaced $r$ by $\frac{r}{m}$, which is the annual rate divided by the number of compounding periods per year, and $t$ by $n$, which is the number of compounding periods.

THE UNPAID BALANCE METHOD FOR COMPUTING THE FINANCE CHARGE ON A CREDIT CARD LOAN This method also uses the simple interest formula $I = Prt$; however,

$P =$ previous month’s balance + finance charge + purchases made − returns − payments.

The variable $r$ is the annual interest rate, and $t = \frac{1}{12}$.

FORMULA FOR FINDING THE FUTURE VALUE OF AN ORDINARY ANNUITY Assume that we are making $n$ regular payments, $R$, into an ordinary annuity. The interest is being compounded $m$ times a year and deposits are made at the end of each compounding period. The future value (or amount), $A$, of this annuity at the end of the $n$ periods is given by the equation

$$A = R\frac{\left(1 + \frac{r}{m}\right)^n - 1}{\frac{r}{m}}.$$ 

FORMULA FOR FINDING PAYMENTS ON AN AMORTIZED LOAN Assume that you borrow an amount $P$, which you will repay by taking out an amortized loan. You will make $m$ periodic payments per year for $n$ total payments and the annual interest rate is $r$. Then, you can find your payment by solving for $R$ in the equation

$$P\left(1 + \frac{r}{m}\right)^n = R\left(\frac{1 + \frac{r}{m}}{\frac{r}{m}} - 1\right)^*.$$ 


<table>
<thead>
<tr>
<th>Method</th>
<th>How the Winning Candidate Is Determined</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plurality</td>
<td>The candidate receiving the most votes wins.</td>
</tr>
<tr>
<td>Borda count</td>
<td>Voters rank all candidates by assigning a set number of points to first choice, second choice, third choice, and so on; the candidate with the most points wins.</td>
</tr>
<tr>
<td>Plurality-with-elimination</td>
<td>Successive rounds of elections are held, with the candidate receiving the fewest votes being dropped from the ballot each time, until one candidate receives a majority of votes.</td>
</tr>
<tr>
<td>Pairwise comparison</td>
<td>Candidates are compared in pairs, with a point being assigned the voters’ preference in each pair. (In the case of a tie, each candidate gets a half point.) After all pairs of candidates have been compared, the candidate receiving the most points wins.</td>
</tr>
</tbody>
</table>

**HAMILTON’S APPORTIONMENT METHOD**

a) Find the standard divisor for the apportionment (total population/total number of representatives).

b) Find the standard quota (state’s population/standard divisor) for each state and round it down to its lower quota. Assign that number of representatives to each state.

c) If there are any representatives left over, assign them to states in order according to the size of the fractional parts of the states’ standard quotas.

**RULE FOR COMPUTING THE PROBABILITY OF A UNION OF TWO EVENTS** If $E$ and $F$ are events, then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

**GENERAL RULE FOR COMPUTING $P(F|E)$** If $E$ and $F$ are events in a sample space, then

$$P(F|E) = \frac{P(E \cap F)}{P(E)}.$$

**FORMULA FOR CONVERTING RAW SCORES TO z-SCORES** Assume a normal distribution has a mean of $\mu$ and a standard deviation of $\sigma$. We use the equation

$$z = \frac{x - \mu}{\sigma}$$

to convert a value $x$ in the nonstandard distribution to a z-score.
1. The truth table can be set up using either of the methods shown below.

Method 1:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p → q</th>
<th>～p</th>
<th>～q</th>
<th>～p ∧ ～q</th>
<th>(p → q) ∨ (～p ∧ ～q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Method 2:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(p → q)</th>
<th>∨</th>
<th>(～p ∧ ～q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

2. We use the formula \( A = R \left[ \frac{\left(1 + \frac{r}{m}\right)^n - 1}{\frac{r}{m}} \right] \) with \( A = 70,000, r = 0.075, m = 12, n = 180. \)

\[
70,000 = R \left[ \frac{\left(1 + \frac{0.075}{12}\right)^{180} - 1}{\frac{0.075}{12}} \right]
\]

\[
70,000 = R \left[ \frac{\left(1.00625\right)^{180} - 1}{0.00625} \right]
\]

\[
70,000 = R \left[ \frac{2.069451727...}{0.00625} \right]
\]

\[
70,000 = R \left[ 331.1122763... \right]
\]

\[ R = 211.41 \]

Thus, the monthly payment amount should be $\text{211.41}. \]
3. (a) Yes, the graph can be traced. Euler’s Theorem states that a graph can be traced if it is connected and has either zero or two odd vertices. Since this graph is connected and has two odd vertices, it can be traced.

(b) For a graph to be Eulerian, it must be connected and have zero odd vertices, so this graph is not Eulerian. To Eulerize the graph, we add an additional edge connecting vertices A and D. Then the possible answers for an Euler circuit beginning at vertex A are ABCADA, ACBADA, ADABCA, or ADACBA.

4. (a) We calculate the standard divisor then set up a table to do the apportionment:

\[ d = \frac{120 + 141 + 39}{11} = \frac{300}{11} = 27.27 \]

<table>
<thead>
<tr>
<th>Building</th>
<th>Standard Quota</th>
<th>Integer Part</th>
<th>Fractional Part</th>
<th>Apportionment</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.40</td>
<td>4</td>
<td>0.40</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>5.17</td>
<td>5</td>
<td>0.17</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>1.43</td>
<td>1</td>
<td>0.43</td>
<td>2</td>
</tr>
<tr>
<td>Totals:</td>
<td>10</td>
<td></td>
<td>0.43</td>
<td>11</td>
</tr>
</tbody>
</table>

Thus, Building A gets 4, Building B gets 5, and Building C gets 2.

(b) The committee will now have 12 members, so we have to calculate the new standard divisor then set up a table to do the apportionment:

\[ d = \frac{300}{12} = 25 \]

Then:

<table>
<thead>
<tr>
<th>Building</th>
<th>Standard Quota</th>
<th>Integer Part</th>
<th>Fractional Part</th>
<th>Apportionment</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.80</td>
<td>4</td>
<td>0.80</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>5.64</td>
<td>5</td>
<td>0.64</td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td>1.56</td>
<td>1</td>
<td>0.56</td>
<td>1</td>
</tr>
<tr>
<td>Totals:</td>
<td>10</td>
<td></td>
<td>0.56</td>
<td>12</td>
</tr>
</tbody>
</table>

Thus, Building A gets 5, Building B gets 6, and Building C gets 1.

(c) Yes, the Alabama Paradox occurs. This is because, with no changes in the number of units per building, Building C lost a committee member to another building when the total number of committee members was increased.
5. Let $E = \{\text{the card is a heart}\}$ and $F = \{\text{the card is a face card}\}$. Then we want to find $P(E \cup F)$. We use the formula $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ where $P(E) = \frac{13}{52}$ since there are 13 hearts in a deck, $P(F) = \frac{12}{52}$ since there are 12 face cards in a deck, and $P(E \cup F) = \frac{3}{52}$ since there are 3 cards that are both hearts and face cards (the jack, queen, and king of hearts). Thus:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}$$

6. We start by putting the data values in order: 1, 2, 4, 5, 6, 6, 7, 8, 9, 10, 11, 12.

**Mean:** To find the mean, we add all the values together, then divide by the total number of data values:

$$\frac{1 + 2 + 4 + 5 + 6 + 6 + 7 + 8 + 9 + 10 + 11 + 12}{12} = \frac{81}{12} = 6.75$$

**Median:** Since there are 12 data values, we use the average of the 6th and the 7th values in the ordered list. The 6th value is 6 and the 7th value is 7. Thus, \(\frac{6 + 7}{2} = 6.5\).

**Mode:** The mode is 6 since that is the data value that occurs most frequently.

7. (a) For the plurality method, we tally up first place votes. The candidate with the most first place votes wins. Thus, $A : 3, B : 7, C : 11$ so candidate **C wins**.

(b) For the Borda count method with 3 candidates, each first place vote will be worth 3 points, each second place vote will be worth 2 points, and each third place vote will be worth 1 point. We calculate how many points each candidate received:

$$A : 3(3) + 2(6) + 1(12) = 33$$

$$B : 3(7) + 2(13) + 1(1) = 48$$

$$C : 3(11) + 2(2) + 1(8) = 45$$

Since candidate $B$ ended up with the most points, candidate **B wins**.

(c) **Yes**, the majority criterion is violated using the Borda count method. The majority criterion says that if a candidate is ranked first by a majority of the voters, then that candidate should win the election. Candidate $C$ was ranked first by a majority of voters, but Candidate $B$ won the election according to the Borda count method.
8. This problem could be solved using a number of problem solving techniques (solving a system of equations, using a visual representation, trial and error, etc.) If we wanted to solve a system of equations, we could let $a$, $b$, and $c$ refer to the number of goals scored by Alex, Becca, and Carly respectively. Then we know that $a + b + c = 35$. Since Becca scored three fewer goals than Alex, we know that $b = a - 3$, and since Carly scored two more goals than Alex, we know that $c = a + 2$. Then we can use substitution to rewrite the first equation only in terms of the variable $a$:

\[
\begin{align*}
  a + b + c &= 35 \\
  a + (a - 3) + (a + 2) &= 35 \\
  3a - 1 &= 35 \\
  3a &= 36 \\
  a &= 12
\end{align*}
\]

Then substituting in $a = 12$, we get that

\[
b = a - 3 = 12 - 3 = 9 \quad \text{and} \quad c = a + 2 = 12 + 2 = 14
\]

Thus, **Alex scored 12 goals, Becca scored 9 goals, and Carly scored 14 goals.**

9. (a) Since the graph is complete and has 5 vertices, you would have to consider

\[
\frac{(5 - 1)!}{2} = \frac{24}{2} = 12
\]

Hamilton circuits if you planned on using the brute force method.

(b) Using the Nearest Neighbor algorithm, we start at vertex A. Then of all the edges connected to A, we travel to a new vertex along the edge that has the smallest weight. We continue doing this, always traveling to new vertices. After all the vertices have been used, we return to vertex A.

Using the given weighted graph, we travel from A to D (weight 4), from D to C (weight 8), from C to E (weight 2), from E to B (weight 7), and from B back to A (weight 10). Thus, using the Nearest Neighbor method produces the Hamilton circuit **ADCEBA**.

(c) We add up the weights from part b: $4 + 8 + 2 + 7 + 10 = 31$. 
10. We look for a counterexample to the conclusion.

Since we were able to correctly represent, “Some animals are dangerous,” and, “All lions are animals,” in such a way that the conclusion is not true, we have found a counterexample. Therefore, the syllogism is invalid.

11. To calculate the finance charge using the unpaid balance method, we use the simple interest formula $I = Prt$ where $r = 0.19$, $t = \frac{1}{12}$, and:

$$P = \text{previous month’s balance} + \text{finance charge} + \text{purchases made} - \text{returns} - \text{payments}$$

$$P = 509 + (509 \cdot 0.19 \cdot \frac{1}{12}) + 89 + 127 - 208$$

$$P = 509 + 8.06 + 89 + 127 - 208$$

$$P = 525.06$$

Then the finance charge is $525.06 \cdot 0.19 \cdot \frac{1}{12} = $8.31.

12. We calculate the average constituency for each state:

State A : $\frac{488,895}{11} = 44,445$

State B : $\frac{325,098}{9} = 36,112$

Since State A has a larger average constituency than State B, State A is more poorly represented.
13. (a) We list the coalitions, their weights, whether or not they are winning, and the critical voters. A coalition is winning if its weight is equal to or greater than the quota, which is 11. A voter in a winning coalition is critical if it is the case that if they were to drop out of the coalition, the coalition would no longer be winning.

<table>
<thead>
<tr>
<th>Coalition</th>
<th>Weight</th>
<th>Winning?</th>
<th>Critical Voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>D</td>
<td>10</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>A,B</td>
<td>7</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>A,C</td>
<td>8</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>A,D</td>
<td>13</td>
<td>yes</td>
<td>A,D</td>
</tr>
<tr>
<td>B,C</td>
<td>9</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>B,D</td>
<td>14</td>
<td>yes</td>
<td>B,D</td>
</tr>
<tr>
<td>C,D</td>
<td>15</td>
<td>yes</td>
<td>C,D</td>
</tr>
<tr>
<td>A,B,C</td>
<td>12</td>
<td>yes</td>
<td>A,B,C</td>
</tr>
<tr>
<td>A,B,D</td>
<td>17</td>
<td>yes</td>
<td>D</td>
</tr>
<tr>
<td>A,C,D</td>
<td>18</td>
<td>yes</td>
<td>D</td>
</tr>
<tr>
<td>B,C,D</td>
<td>19</td>
<td>yes</td>
<td>D</td>
</tr>
<tr>
<td>A,B,C,D</td>
<td>22</td>
<td>yes</td>
<td>none</td>
</tr>
</tbody>
</table>

(b) The Banzhaf power index for each voter is:

\[
A : \frac{2}{12} = \frac{1}{6}, \quad B : \frac{2}{12} = \frac{1}{6}, \quad C : \frac{2}{12} = \frac{1}{6}, \quad D : \frac{6}{12} = \frac{1}{2}
\]

14. A game is fair if the expected value of playing the game is zero. We consider the values and probabilities associated with each possible outcome.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Win</td>
<td>$36</td>
<td>\frac{2}{6} = \frac{1}{3}</td>
</tr>
<tr>
<td>Lose</td>
<td>$-36</td>
<td>\frac{4}{6} = \frac{2}{3}</td>
</tr>
</tbody>
</table>

Then we can calculate the expected value: \((\frac{1}{3} \cdot 36) + (\frac{2}{3} \cdot (-36)) = -$12. Since the expected value of playing the game is not zero, the game is **not fair**.