TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
August 2003

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF ($T_2$). GIVE A PRECISE STATEMENT OF ANY MAJOR THEOREM REFERENCED IN ANY ARGUMENT. GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE.

1.) Let $U = \{ U_\alpha \mid \alpha \in A \}$ be an open cover of the compact metric space $(X, d)$. Show that there exists a number $\delta > 0$ such that for every subset $H$ of $X$ with $\text{diam}(H) < \delta$ there exists $\alpha_0 \in A$ such that $H \subset U_{\alpha_0}$.

2.) Let $(X, d)$ be a metric space. Show that the following are equivalent.
   
   a.) $X$ has a countable dense subset.
   
   b.) $X$ has a countable basis for its topology.
   
   c.) Every open cover of $X$ has a countable subcover.

3.) Let $X = \prod_{\alpha \in A} X_\alpha$, where $A$ is an arbitrary indexing set and each $X_\alpha$ is nonempty. Show that $X$ is regular if and only if each $X_\alpha$ is regular.

   Give an example to show that the product of normal spaces need not be normal. Clearly indicate why your example has the desired properties.

4.) Show that if $f : X \rightarrow Y$ is a closed, continuous surjection with $X$ locally compact and each $f^{-1}(y)$ compact, then $Y$ is locally compact.

   Show that if the hypothesis that each $f^{-1}(y)$ is compact is omitted then $Y$ need not be locally compact.

5.) Let $X = \prod_{\alpha \in A} X_\alpha$, where $A$ is an arbitrary indexing set and each $X_\alpha$ is nonempty. Prove that $X$ is connected if and only if each $X_\alpha$ is connected.

6.) Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map of the path connected space $B$. Show that if $p^{-1}(b_0)$ has exactly $k$ elements, then $p^{-1}(b)$ has exactly $k$ elements for each $b \in B$.

7.) Let $h : S^1 \rightarrow S^1$ be a nullhomotopic continuous function from the unit circle $S^1$ to itself. Show that $h$ has a fixed point and that $h$ maps some point $x \in S^1$ to its antipode $-x$.

8.) Assume that each of $X_1$, $X_2$ and $X_1 \cap X_2$ is an arcwise-connected open subset of the space $X$, where $X = X_1 \cup X_2$ and $x_0 \in X_1 \cap X_2$. Let $i : X_1 \rightarrow X$ and $j : X_2 \rightarrow X$ be the inclusion mappings of $X_1$ and $X_2$, respectively, into $X$. Show that the images of the induced homomorphisms $i_* : \pi_1(X_1, x_0) \rightarrow \pi_1(X, x_0)$ and $j_* : \pi_1(X_2, x_0) \rightarrow \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$. (This is a major step in the proof of the Seifert - van Kampen theorem. Do not quote this theorem as part of the above argument.)

   Using this result, give a presentation for the fundamental group of the surface represented by the two-holed torus.