Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let \( \mu^* \) be an outer measure on \( X \). A collection of subsets \( \{E_1, E_2, \ldots\} \) of \( X \) is called a partition of \( X \) if \( E_i \cap E_j = \emptyset \), for any \( i \neq j \), and \( \bigcup_{i=1}^{\infty} E_i = X \). Prove that all the subsets in a partition \( \{E_n\}_{n=1}^{\infty} \) are \( \mu^* \)-measurable if and only if \( \mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i) \) for any subset \( A \) of \( X \).

2. Let \( S \) be a dense subset of \( \mathbb{R} \) and \( (X, \mathcal{M}) \) be a measurable space. Prove that a real valued function \( f \) is measurable if and only if \( \{x : f(x) \leq r\} \in \mathcal{M} \) for all \( r \in S \).

3. Let \( E \) be a Lebesgue measurable subset of \( \mathbb{R} \) with \( m(E) < \infty \). Prove that for any \( \epsilon > 0 \), there is a finite disjoint union of open intervals \( A \) such that \( m(E \setminus A) = m((E \setminus A) \cup (A \setminus E)) < \epsilon \).

4. Let \( (X, \mathcal{M}, \mu) \) be a measure space, and \( \{f_n, n = 1, 2, \ldots\} \) be a sequence of measurable functions which converges a.e. Prove that if there exists a subsequence \( \{n_k\} \) such that \( \lim_{k \to \infty} \int |f_{n_k}| \, d\mu = 0 \), then \( \lim_{n \to \infty} f_n(x) = 0 \) a.e.

5. Let \( \{f_n\} \) be a sequence of measurable functions over a \( \sigma \)-finite measure space \( (X, \mathcal{M}, \mu) \). Prove that if \( \sum_{n=1}^{\infty} |f_n| \) is integrable, then each \( f_n \) is integrable. \( \sum_{n=1}^{\infty} f_n \) converges almost everywhere and is integrable, and

\[
\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.
\]

6. Let \( -\infty < a < b < \infty \) and \( f \) be a function of bounded variation on \([a, b]\). Prove that \( f \) can be written as \( f = g + h \), where \( g \) is absolutely continuous and \( h' = 0 \) a.e. on \([a, b]\).

7. Let \( (X, \mathcal{M}) \) be a measurable space, \( \nu \) be a signed measure on \( (X, \mathcal{M}) \), \( X = P \cup N \) be a Hahn decomposition for \( \nu \). For any \( E \in \mathcal{M} \), prove that

\[
|\nu(E)| = |\nu|(E) \quad \text{if and only if} \quad |\nu|(E \cap P) = 0 \quad \text{or} \quad |\nu|(E \cap N) = 0.
\]

8. Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be two norms on a vector space \( X \) over \( \mathbb{R} \). Prove that if there is a \( c > 0 \) such that \( \|x\|_1 \leq c \|x\|_2 \) for all \( x \in X \), and \( X \) is complete with respect to both norms, then the two norms are equivalent.

9. Let \( H \) be an infinite dimensional Hilbert space. Prove that the unit sphere \( S = \{f \in H : \|f\| = 1\} \) contains a sequence that converges to 0 weakly (Simply using the fact that \( S \) is weakly dense in the unit ball is not acceptable).

10. Let \( (X, \mathcal{M}, \mu) \) be a measure space, and \( f \in L^p \cap L^\infty \) for some \( 1 \leq p < \infty \) with \( \|f\|_\infty > 0 \). For any \( 0 < \alpha < 1 \), let

\[
E_\alpha = \{x : |f(x)| > \alpha \|f\|_\infty\}.
\]

Prove that

\[
0 < \mu(E_\alpha) < \infty, \quad \text{and} \quad \|f\|_p \geq \alpha \|f\|_\infty \mu(E_\alpha)^{1/p}.
\]