Real Analysis Preliminary Examination
August, 2010

Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let \( \mu \) be a complete measure. Prove that if \( f \) is a measurable function and if \( f = g \) \( \mu \)-a.e., then \( g \) is a measurable function.

2. Let \( E \subset \mathcal{P}(X) \) and \( \rho : E \to [0, \infty] \) be such that \( \emptyset \in E, X \in E, \text{ and } \rho(\emptyset) = 0. \forall A \subset X, \) define

\[
\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) \mid E_j \in E \text{ and } A \subset \bigcup_{j=1}^{\infty} E_j \right\}.
\]

Prove that \( \mu^* \) is an outer measure.

3. Let \( f \) be a nonnegative element of \( L^1[0,1] \). Prove that

\[
\lim_{n \to \infty} \int_0^1 (f(x))^{1/n} \, dx = m(\{x \in [0,1] \mid f(x) > 0\}).
\]

4. Suppose \( \{f_n\} \subset L^+(\text{nonnegative measurable functions}), f_n \to f \) pointwise, and \( \int f = \lim_{n \to \infty} \int f_n < \infty. \) Prove that for all \( E \in \mathcal{M} \)

\[
\int_E f = \lim_{n \to \infty} \int_E f_n.
\]

5. Let \( 1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1, f \in L^p(\mathbb{R}), \text{ and } g \in L^q(\mathbb{R}). \) Prove that

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy
\]

exists for every \( x, \|f \ast g\|_{\infty} \leq \|f\|_p \|g\|_q, \text{ and } f \ast g \text{ is uniformly continuous}.
\]

6. Suppose \( f \) is absolutely continuous on \( \mathbb{R} \) and \( f \in L^1(\mathbb{R}). \) Prove that if, in addition,

\[
\lim_{t \to 0^+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| \, dx = 0,
\]

then \( f \equiv 0. \) (Hint: begin your work with Fatou)

7. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be Lebesgue measurable. Assuming that Lebesgue measure is translation invariant, prove that the Lebesgue integral is translation invariant, i.e. prove that if \( f \in L^1(\mathbb{R}^n) \), then

\[
\int f(x) \, dm^n = \int f(x + y) \, dm^n.
\]

8. Suppose that there exists a \( p < \infty \) such that \( f \in L^q \cap L^\infty \) for all \( q \geq p. \) Prove that

\[
\|f\|_{\infty} = \lim_{q \to \infty} \|f\|_q.
\]

(Warning: Your argument should also show that the limit exists.)

9. Let \( X \) and \( Y \) be normed vector spaces and \( T \) be a bounded linear transformation from \( X \) to \( Y. \) Define \( S : Y^* \to X^* \) by \( S(f) = f \circ T. \) Prove that \( S \) is a bounded linear transformation and that \( \|S\| = \|T\|. \) (Possible hint: Hahn-Banach.)