Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let $f$ be a nonnegative function in $L^1(m)$ where $m$ is the Lebesgue measure on the real line $\mathbb{R}$. Show that the function $g$ defined by
   \[ g(x) := \int_{(-\infty, x]} f \, dm \]
   is a continuous function.

2. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed vector spaces. Define what it means for a linear map from $X \to Y$ to be bounded. Let $L(X,Y)$ be the set of bounded linear maps from $X$ to $Y$. Define the operator norm on $L(X,Y)$ and show that if $(Y, \|\cdot\|_2)$ is a Banach space then $L(X,Y)$ is a Banach space (with the operator norm).

3. Let $\{u_\alpha\}_{\alpha \in A}$ be an indexed orthonormal set in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is conjugate linear in the second slot. Show that the following are equivalent:
   a) If $\langle x, u_\alpha \rangle = 0$ for all $\alpha \in A$ then $x = 0$.
   b) For each $x \in \mathcal{H}$, $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ where this sum has only countably many nonzero terms and the convergence is in the norm derived from $\langle \cdot, \cdot \rangle$ and independent of the order of the nonzero terms.
   (You may use Bessel’s Inequality.)

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that $g$ is a non-negative measurable function on $X$. Show that
   a) $\nu(E) := \int_E g \, d\mu$ for $E \in \mathcal{M}$ defines a measure on $X$ (with domain $\mathcal{M}$).
   b) For a non-negative measurable function on $X$ we have $\int_X f \, d\nu = \int_X f \, g \, d\mu$.

5. Suppose that $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfies the following conditions:
   a) $f$ is bounded.
   b) For each $x$, the map $t \mapsto f(x,t)$ is measurable.
   c) The partial derivative $\frac{\partial f}{\partial t}$ exists everywhere and is bounded.
   Show that
   \[ \frac{d}{dt} \int_0^1 f(x,t) \, dx = \int_0^1 \frac{\partial f}{\partial t}(x,t) \, dx \]
   where the integral is the Lebesgue integral.

6. Let $X$ be a Banach space. Show that
   \[ \|x\| = \sup \{ \|\phi(x)\| : \phi \in X^* \text{ with } \|\phi\| = 1 \}. \]

7. Let $g : [a, b] \to \mathbb{R}$ be absolutely continuous. Show that if $E \subset [a, b]$ has Lebesgue measure zero then $g(E)$ also has Lebesgue measure zero.

8. Prove the Riemann-Lebesgue lemma. If $f \in L^1(\mathbb{R})$ then
   \[ \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx = 0 \]
   (Here we are using Lebesgue integration on the line).

9. Let $m$ be the Lebesgue measure on the real line $\mathbb{R}$. Show by example that there exists a sequence of functions in $L^1(m)$ that converges to zero in $L^1(m)$ but such that the sequence does not converge to zero pointwise almost everywhere.

10. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces. Suppose that $K \in L^2(X \times Y, \mu \otimes \nu)$. Show that if $f \in L^2(Y, \nu)$ then the formula
    \[ (Tf)(x) = \int K(x,y)f(y) \, d\nu(y) \]
    defines a function $Tf \in L^2(X, \mu)$. Show that $T : L^2(X, \mu) \to L^2(Y, \nu)$ is a bounded linear operator satisfying $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$.  