Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let \((H, \langle \ , \, \rangle)\) be a Hilbert space and \(A : H \to H\) be a bounded linear operator. Show that there is a unique bounded linear operator \(A^* : H \to H\) such that \(\langle Ax, y \rangle = \langle x, A^* y \rangle\) for all \(x, y \in H\). Show that \(\|A^*\| = \|A\|\).

2. Prove the following theorem (Egoroff’s Theorem):

\textbf{Theorem:} Let \((X, \mu)\) be a measure space with \(\mu(X) < \infty\). Suppose that \(f, f_n, n = 1, 2, \ldots\) are measurable functions such that \(f_n \to f\) a.e. Then for every \(\epsilon > 0\) there exists a measurable set \(E \subset X\) such that \(\mu(E) < \epsilon\) and such that \(f_n \to f\) uniformly on \(E^c = X \setminus E\).

3. Let \(m\) be Lebesgue measure on \(\mathbb{R}\). Show that for \(f \in L^p(m), p \geq 1,\) we have that \(f_h \in L^p(m)\) where \(f_h(x) := f(x + h)\). Show that \(\lim_{h \to 0} \|f - f_h\| = 0\).

4. Show that if \((f_n)_{n=1}^\infty\) is a sequence of measurable extended real valued functions on a measure space \(X\), then the set \(\{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists} \}\) is measurable.

5. For \(t \in \mathbb{R}\), let \([t]\) denote the greatest integer not exceeding \(t\). Let \(F(t) := t + [t]\). Find

\[
\int_0^\infty e^{-t} dF.
\]

6. State and prove the Closed Graph Theorem

7. Compute \(\lim_{n \to \infty} \int_0^\infty \frac{x^n}{1 + x^n} dx\). Justify all steps.

8. Give an example of measure spaces \((X, M, \mu), (Y, N, \nu)\), and a non-negative measurable function \(f\) defined on \(X \times Y\) such that

\[
\int_Y \left( \int_X f(x, y) \, d\mu(x) \right) \, d\nu(y) \neq \int_X \left( \int_Y f(x, y) \, d\nu(y) \right) \, d\mu(x).
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