Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Let 
\[ g(x) = \sum_{n=1}^{\infty} \frac{\chi_{[n,n+1)}(x)}{n^2}, \]
and for Lebesgue measurable sets \( E \) define \( \nu E = \int_E g(x) \, dx \). Find \( \int_{\mathbb{R}} 1 \, d\nu \) and \( \int_{\mathbb{R}} x \, d\nu \).

2. Suppose \( f(x) \) is Lebesgue measurable on \([0, 1]\). Show that \( g(x, y) = f(x) - f(y) \) is measurable on \([0, 1] \times [0, 1]\) with respect to the two-dimensional Lebesgue measure.

3. Let \( f \) be a continuous real-valued function on the unit interval \([0, 1]\). Show that for each \( \epsilon > 0 \) there exists a nonnegative integer \( n \) and \( c_0, \ldots, c_n \in \mathbb{R} \) so that
\[ |c_0 + c_1 e^{-x} + c_2 e^{-2x} + \cdots + c_n e^{-nx} - f(x)| < \epsilon \]
for all \( x \in [0, 1] \).

4. Consider the function \( h(x) \) defined by
\[ h(x) = \begin{cases} -x^2, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{Z} \setminus \{0\} \\ x^2, & \text{otherwise.} \end{cases} \]
Is \( h(x) \) of bounded variation on \([0, 1]\)? (Prove your answer, of course!)

5. Let \((X, \mathcal{B}, \mu)\) be a finite measure space with \( \mu X = 1 \). If \( E_1, E_2, \ldots, E_{16} \) are measurable sets with \( \mu E_j = 1/3 \) for each \( j \), show that for some \( 1 \leq j_1 < j_2 < j_3 < j_4 < j_5 < j_6 \leq 16 \), \( \mu(E_{j_1} \cap \cdots \cap E_{j_6}) > 0 \).

6. For \( f, g \in L^1(\mathbb{R}) \) the convolution \( f * g \) is defined by \( (f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t) \, dt \). For \( f \in L^1(\mathbb{R}) \), the Fourier transform \( \hat{f} \) of \( f \) is defined by \( \hat{f}(s) = \int e^{ist} f(t) \, dt \). Show that \( \hat{f} \) is a bounded complex function and \( \hat{f} * g = \hat{f} \hat{g} \).

(Recall: If \( F_1, F_2 \) are integrable real-valued functions, the integral of the complex-valued function \( F = F_1 + iF_2 \) is \( \int F = \int F_1 + i \int F_2 \).)

7. (Riemann-Lebesgue Theorem) If \( f \) is integrable on \( \mathbb{R} \), show that
\[ \lim_{k \to \infty} \int_{\mathbb{R}} f(x) \cos(kx) \, dx = 0. \]

8. Find
\[ \lim_{n \to \infty} \int_{a}^{\infty} \frac{n}{1 + n^2x^2} \, dx \]
for \( a > 0, a = 0 \) and \( a < 0 \). Justify all of your steps!

9. State and prove Fatou’s Lemma and give an example in which strict inequality occurs.

10. Suppose \( f \) and its derivative \( f' \) are absolutely continuous on \([0, 1]\) with \( f' \) increasing. Set \( g(x, y) = f''(x + y)\chi_{[0,1]}(x+y) \) and show that
\[ \int_{[0,1] \times [0,1]} g(x, y) \, d(x \times y) = f'(1) + f(0) - f(1). \]
(Justify your steps.)