Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Let $\mu$ be a finite measure on $\mathbb{R}$ absolutely continuous with respect to Lebesgue measure, and $g \in L^\infty(\mu)$. Show that the function $G(t) = \int_{\mathbb{R}} g \chi_{[0,t]} \, d\mu$ is continuous on $\mathbb{R}$. (Recall that for $a, b \in \mathbb{R}$, $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$.)

2. Give the definition of Lebesgue integral and use it to show that if $f$ is Lebesgue integrable on $[a, b]$ and $c > 0$ and $d \in \mathbb{R}$ then $\int_b^a f(x) \, dx = c \int_{b-c}^{a-c} f(cx + d) \, dx$.

3. If $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$, show that $fg \in L^1[0,1]$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$. Give an example for which equality holds, and an example for which strict inequality holds.

4. a. Prove the ‘Parallelogram Law’ for Hilbert spaces: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

   b. Show that the Banach space $C[0,1]$ with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$ does not admit an inner product.

5. Suppose $f$ is Lebesgue integrable on $[0,1]$.
   a. If $f$ is continuous on $[0,1]$ and $\int_0^1 |f| \, dx = 0$, show that $f$ is identically zero on $[0,1]$.
   b. Show that the previous statement is false without the continuity assumption.
   c. Define precisely $L^1[0,1]$ and show that $\|g\|_1 = \int_0^1 |g| \, dx$ defines a norm on $L^1[0,1]$ (be sure to resolve the seeming contradiction with Part b!).

6. Show that a continuous real-valued function on $(0,1)$ is the uniform limit of a sequence of polynomials on $(0,1)$ if and only if $f$ can be extended to a continuous function on $[0,1]$.

7. Let $m^*$ denote Lebesgue outer measure on $\mathbb{R}^2$. If $p(x, y)$ is a nonzero polynomial and $Z = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\}$, show that $m^*Z = 0$.

8. Prove that the Radon-Nikodym derivative, when it is defined, is unique modulo sets of measure zero.

9. Let $\mathcal{B}$ denote the Borel subsets of $\mathbb{R}$ and $m$ denote Lebesgue measure on $\mathcal{B}$. For $E \in \mathcal{B}$ define $\nu_E = \sum_{n=0}^\infty \frac{m(E \cap [n, n+1])}{2^n}$. Show that $(\mathbb{R}, \mathcal{B}, \nu)$ is a finite measure space and find the Radon-Nikodym derivative $\frac{d\nu}{dm}$.

10. Prove or disprove by counterexample: If $g(t)$ is continuous on $[0,1]$, then there is a subinterval $(a, b) \subset [0,1]$ with $a < b$ for which $g$ is monotonic on $(a, b)$. 