Real Analysis Preliminary Examination
2003

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Consider the set $X = \{a, b, c\}$. Starting with the map $\phi$ given by

\[
\begin{align*}
\phi(\emptyset) &= 0, \\
\phi(\{a\}) &= 1, \\
\phi(\{a, b\}) &= 2, \\
\phi(\{a, b, c\}) &= 5,
\end{align*}
\]

and using the Carathéodory extension, construct an outer measure $\mu^*$ on $X$. Is the set $\{a, b\}$ $\mu^*$-measurable?

2. Let $F(x) = 3x + \lfloor \sqrt{x} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function of $x$. Denote by $\mu_F$ the Lebesgue-Stieltjes measure associated with $F$. Compute the integral

$$
\int_{[0, 9]} 2^x d\mu_F.
$$

3. Prove the formula

$$
\frac{1}{a^3} = \frac{1}{2} \int_0^\infty x^2 e^{-ax} dx.
$$

Using this formula, rigorously derive

$$
\sum_{n=1}^\infty \frac{1}{n^3} = \frac{1}{2} \int_0^\infty \frac{x^2}{e^x - 1} dx
$$

4. Show that

$$
\int_0^\infty e^{-sx} x^{-1} \sin^2(x) dx = \frac{1}{4} \ln(1 + 4s^{-2}), \quad s > 0
$$

by integrating $e^{-sx} \sin(2xy)$ on $[0, \infty) \times [0, 1]$. Be sure that you verify the hypothesis of the theorems that you use (Hint: A half-angle formula might be useful.)

5. Let $F, G, H : [0, \infty) \to \mathbb{R}$, $F(x) = x + \lfloor x \rfloor$, $G(x) = \lfloor 2x \rfloor$, $H(x) = x^3$, where $\lfloor x \rfloor$ is the greatest integer function, and let $\mu_F, \mu_G$ and $\mu_H$ be the Lebesgue-Stieltjes measures they determine. Compute the Radon-Nikodym derivatives

$$
\frac{d\mu_F}{d\mu_G}, \quad \frac{d\mu_G}{d\mu_H}, \quad \frac{d\mu_H}{d\mu_F}
$$

if they exist. If not, explain why they do not exist.

6. Give an example of a function that is in $L^5(\mathbb{R})$ but not in $L^7(\mathbb{R})$. Give an example of a function that is in $L^7(\mathbb{R})$ but not in $L^5(\mathbb{R})$. 

1
7. Prove that in any Hilbert space, the following polarization formula holds

\[ <x, y> = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right) \].

8. Let \( K(x, t) \in L^2(\mathbb{R}^2) \). Define

\( (Tf)(x) = \int_{\mathbb{R}} K(x, t) f(t) dt \).

Prove that \( T \) is a bounded linear operator from \( L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \).

9. Let \( f(x) = \frac{1}{2} - x \) on the interval \([0, 1)\), and extend \( f \) to be periodic on \( \mathbb{R} \).

a. Find the Fourier series of \( f \).

b. Deduce the formula

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \].

10. Prove that for any real number \( y \),

\[ \sum_{m=-\infty}^{\infty} e^{-2\pi(m+y)^2} = \frac{\sqrt{2}}{2} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/2} e^{2\pi i m y} \].