Real Analysis Preliminary Examination
August 2003

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. All theoretical results that you use must be mentioned.

1. Find the $\sigma$-algebra generated by the subsets $\{1, 2\}$ and $\{2, 3\}$ of the space $X = \{1, 2, 3, 4, 5, 6\}$.

2. Let $(X, \mathcal{M})$ be a measurable space. Prove that if $f : X \to \mathbb{R}$ has the property that $f^{-1}((r, \infty))$ is measurable for any rational number $r$, then $f$ is a measurable function.

3. Rigorously compute the integral

\[ \int_0^1 \ln x \ln(1 - x)\,dx \]

(Hint. Expand in Taylor series the function $\ln(1 - x)$.)

4. Show that

\[ \int_0^\infty x^ne^x\,dx = n! \]

by differentiating the equation

\[ \int_0^\infty e^{-tx}\,dx = \frac{1}{t}. \]

Be sure that you verify the hypothesis of the theorems that you use.

5. Let $F, G : (0, 10) \to \mathbb{R}$, $F(x) = x^3 + \lfloor 3x \rfloor$ and $G(x) = x^5 + \lfloor 2x \rfloor$ and let $\mu_F$ and $\mu_G$ be the Lebesgue-Stieltjes measures they induce on the interval $(0, 10)$. Find the Lebesgue-Radon-Nikodym decomposition of $\mu_F$ with respect to $\mu_G$.

6. Let $X$ be a Banach space and $T \in L(X)$ a linear operator with $\|T\| < 1$. Prove that $I - T$ is invertible. (Hint. Find a series that converges to $(I - T)^{-1}$.)

7. Give an example of a function $f$ with the property that $f \in L^p((2, \infty))$ if and only if $p \geq 2$.

8. Find, with proof, a constant $C$ such that

\[ \left( \int_0^2 f(x)\,dx \right)^5 \leq C \int_0^2 f^5(x)\,dx \]

for all functions $f$ that are positive and continuous on the interval $[0, 2]$.

9. Find the Fourier transform of the one variable function $f(x) = x^3e^{-2\pi x^2}$.

10. Prove that for every $0 < x < 2$ the following formula is valid

\[ \frac{x}{2} = \frac{\pi}{2} - \frac{\sin x}{1} - \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \cdots. \]