Real Analysis Preliminary Examination

2001

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Notations: \( \mathbb{C} \) = the set of complex numbers, \( \mathbb{R} \) = the set of real numbers.

1. Let \( S = \{E_1, E_2, \ldots, E_n\} \) be a collection of nonempty subsets of \( X \) such that

\[
\bigcup_{i=1}^{n} E_i = X \text{ and } E_i \cap E_j = \emptyset \text{ for } i \neq j.
\]

Find the \( \sigma \)-algebra generated by \( S \).

2. For \( x \in \mathbb{R} \), let \( \lfloor x \rfloor \) be the largest integer less than or equal to \( x \). Let

\[
F(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
\lfloor x \rfloor, & \text{if } x > 0
\end{cases},
\]

and let \( \mu_F \) be the Lebesgue-Stieltjes measure associated to \( F \). Compute

\[
\int 3^{-x} \, d\mu_F.
\]

3. Let \( B_\mathbb{R} \) be the Borel \( \sigma \)-algebra on \( \mathbb{R} \), and \( \mu \) be a measure on \( B_\mathbb{R} \) which is finite on every bounded set in \( B_\mathbb{R} \). Define

\[
F(x) = \begin{cases} 
\mu((0, x]), & \text{if } x \geq 0 \\
-\mu((x, 0]), & \text{if } x < 0
\end{cases}
\]

Show that

a. \( F \) is increasing,

b. \( F \) is right continuous,

c. \( \mu \) is the Lebesgue-Stieltjes measure associated to \( F \).

4. Let \( f \) be a nonnegative measurable function on a measure space \( (X, \mathcal{M}, \mu) \), and \( E_1 \subset E_2 \subset \cdots \) be measurable subsets of \( X \). Prove that

\[
\int_{\bigcup_{n=1}^{\infty} E_n} f \, d\mu = \lim_{n \to \infty} \int_{E_n} f \, d\mu.
\]
5. Let \( f_n, n = 1, 2, \ldots, \) be a sequence of integrable functions on a measure space \( (X, \mathcal{M}, \mu) \) such that
\[
\int |f_n| \, d\mu \leq M < \infty \text{ for all } n
\]
and \( f_n \rightarrow f \) in measure. Prove that \( f \) is integrable and
\[
\int |f| \, d\mu \leq M.
\]

6. Show that if a real series \( \sum_{i,j=1}^{\infty} a_{ij} \) converges absolutely, then
\[
\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.
\]

7. Let \( \nu \) be a signed measure on a measurable space \( (X, \mathcal{M}) \), and \( X = P \cup N \) be a Hahn decomposition for \( \nu \). Prove that
   a. \( |\nu(E)| \leq |\nu|(E) \) for all \( E \) in \( \mathcal{M} \),
   b. \( |\nu(E)| = |\nu|(E) \) if and only if either \( \nu(E \cap P) = 0 \) or \( \nu(E \cap N) = 0 \).

8. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces, \( T : \mathcal{X} \rightarrow \mathcal{Y} \) be an injective bounded linear map, and \( \mathcal{M} \) be the range of \( T \). Prove that \( T : \mathcal{X} \rightarrow \mathcal{M} \) is an isomorphism if and only if \( \mathcal{M} \) is closed.

9. Show that in an inner product space over \( \mathbb{C} \),
\[
\langle x, y \rangle = \frac{1}{4} (||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2)
\]
and use it to prove that there is at most one inner product which generates the same induced norm, namely \( ||x|| = \sqrt{\langle x, x \rangle} \).

10. Let \( k(x, t) \) be a Lebesgue measurable function on \( \mathbb{R}^2 \) such that
\[
(\int \int |k(x, t)|^q \, dt \, dx)^{1/q} < \infty, \text{ for } 1 < q < \infty
\]
and let \( p \) satisfy
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
Define
\[
T(f)(x) = \int k(x, t) f(t) \, dt.
\]
Prove that \( T(f) \) is in \( L^q(\mathbb{R}) \) for every \( f \in L^p(\mathbb{R}) \), \( T \) is a bounded linear operator from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) and
\[
||T|| \leq (\int \int |k(x, t)|^q \, dt \, dx)^{1/q}.
\]