Statistics Prelim, August 2010

Work all 7 problems. Begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed. State any theorem or fact you use. You may need the following probability distributions for problems:

\[ \text{Poisson}(\lambda) : f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \ldots, \lambda > 0 \]

\[ \text{Exp}(\lambda) : f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0 \]

\[ b(n, p) : f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n, \quad 0 < p < 1 \]

\[ N(\mu, \sigma^2) : f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right), \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma^2 > 0 \]

\[ \chi^2(d) : f(x) = \frac{1}{\Gamma(d/2)2^{d/2}} x^{d/2-1} e^{-x/2}, \quad x > 0, \quad d > 0 \]

where \( \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx \)

1. (15 points) Let \( Z_1 \) and \( Z_2 \) be independent \( \text{Exp}(\lambda) \) random variables, \( \lambda > 0 \). Define \( X = Z_2 \) and \( Y = Z_1 + Z_1 Z_2 \).

   (a) Find the joint density of \( X \) and \( Y \).

   (b) Find \( E(Y | X = x) \).

   (c) Find \( \text{Var}(E(Y | X)) \).

2. (15 points) Suppose \( X_1, \ldots, X_n \) is a random sample from the probability density function

\[ f(x; \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0. \]

Let \( W_i = -\log(X_i) \) and let \( \theta \) be the unknown parameter.

   (a) Show that \( \sum_{i=1}^{n} W_i \) is a complete and sufficient statistic for \( \theta \).

   (b) Show that the distribution of \( 2\theta \sum_{i=1}^{n} W_i \) is \( \chi^2(2n) \).

   (c) Find the MVUE of \( \theta \). (Hint: Calculate \( E \left( \left( \sum_{i=1}^{n} W_i \right)^{-1} \right) \).

3. (15 points) Let \( (X_1, Y_1), \ldots, (X_n, Y_n) \) be independent copies of \( (X, Y) \), whose joint distribution is specified as follows: the marginal distribution of \( X \) is \( \text{Poisson}(\lambda) \), and conditioning on \( X = x \), \( Y \) is distributed as \( b(x+1, p) \).

   (a) Show that the covariance between \( X \) and \( Y \) is \( \alpha = \rho \lambda \).

   (b) Find the maximum likelihood estimator of \( \alpha \), call it \( \hat{\alpha} \).

   (c) Find the asymptotic distribution of \( \sqrt{n}(\hat{\alpha} - \alpha) \).

4. (10 points) If \( X_n \sim b(n, 1/n) \), show that the limiting distribution of \( X_n \) is \( \text{Poisson}(1) \).