Statistics Prelim, May 2009

Work all 6 problems. Please begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed. You may need the following probability distributions for problems.

- \( b(n, p) \)
  \[ f(x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n \]

- \( \text{Poisson}(\lambda) \)
  \[ f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \ldots \]

- \( N(\mu, \sigma^2) \)
  \[ f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty \]

- \( \text{Exp}(\lambda) \)
  \[ f(x) = \lambda e^{-\lambda x}, \quad x > 0 \]

1. Let \( Y \) be a random variable distributed as \( \text{Poisson}(\lambda) \) and suppose that the conditional distribution of the random variable \( X \), given \( Y = y \), is \( b(y, p) \).
   (a) Find the marginal probability mass function of \( X \).
   (b) Find the conditional probability mass function of \( Y \) given \( X \).

2. Suppose \( X_1, X_2 \) and \( X_3 \) are IID random variables from \( \text{Exp}(\lambda) \). Let \( Z_1 = 3Y_1, Z_2 = 2(Y_2 - Y_1) \) and \( Z_3 = Y_3 - Y_2 \), where \( Y_1 < Y_2 < Y_3 \) are the order statistics corresponding to \( X_1, X_2, X_3 \).
   (a) Find the joint pdf of \( (Z_1, Z_2, Z_3) \).
   (b) Show that \( Z_1, Z_2 \) and \( Z_3 \) are independent and find their marginal distributions.

3. Suppose \( Y_i \sim N(\beta x_i, 1) \) where the \( x_i \)'s are known constants and \( \beta \) is an unknown parameter. Assume the \( Y_i \)'s are independent.
   (a) Find the MLE \( \hat{\beta} \) of \( \beta \).
   (b) Show that \( \hat{\beta} \) is an unbiased estimator of \( \beta \)
   (c) Find the MSE of \( \hat{\beta} \).
4. Let $X_1, \ldots, X_n$ be a sample from a population with the density

$$f(x; \theta) = \theta(\theta + 1)x^{\theta-1}(1-x),$$

where $0 < x < 1$, $\theta > 0$.

(a) Show that

$$T_n = \frac{2\bar{X}}{1 - \bar{X}}$$

is a method of moments estimate of $\theta$.

(b) Show that

$$\sqrt{n}(T_n - \theta) \overset{d}{\to} N(0, \theta(\theta + 2)^2/2(\theta + 3))$$

(c) Show that $T_n$ is not asymptotically efficient.

5. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu_1, \sigma^2)$ and let $Y_1, \ldots, Y_m$ be a random sample from $N(\mu_2, 4\sigma^2)$.

(a) Find the maximum likelihood estimators of $\mu_1, \mu_2,$ and $\sigma^2$.

(b) Derive the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.

(c) What is the sampling distribution of your test statistic in (b) under $H_0$?

6. Suppose that $X_1, \ldots, X_n$ is a random sample from $Exp(\theta)$ and that $Y_1, \ldots, Y_m$ is a random sample from $Exp(\mu)$. Assume $X_i$ and $Y_j$ are independent for any $i$ and $j$.

(a) Construct the likelihood ratio test that depends on the statistic

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{j=1}^{m} Y_j}$$

for testing $H_0 : \theta = \mu$ against $H_1 : \theta \neq \mu$.

(b) Find the distribution of $T$ under $H_0$.

(c) Suppose that $n = m = 1$. Give the exact rejection region for the size 0.05 LRT for testing $H_0 : \theta = \mu$ against $H_1 : \theta \neq \mu$. 

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