Work all 8 problems. Begin each problem on a new page, using one side of the sheet. Throughout “p.d.f.” means “probability density function”, $\mathbb{R} = (-\infty, \infty)$, and $1_A$ is the indicator function of the set $A$ (i.e. $1_A(x) = 1$ if $x \in A$, 0 otherwise). A table of the standard normal distribution is attached.

1. Suppose that we are given a random sample $X_1, \ldots, X_n$ from the p.d.f.
   \[ f_\theta(x) = \theta x^{\theta-1} 1_{(0,1)}(x), \quad x \in \mathbb{R}, \]
   where $\theta > 0$ is an unknown parameter. The null hypothesis $H_0 : \theta = 1$ is to be tested against the alternative $H_1 : \theta > 1$.
   \[ \text{a.} \quad \text{Determine the family of uniformly most powerful tests.} \]
   \[ \text{b.} \quad \text{Assuming that the sample size } n \text{ is sufficiently large, use the central limit theorem to find a uniformly most powerful test of approximate significance level } \alpha = 0.05. \]
   \[ \text{c.} \quad \text{For sample size } n = 1 \text{ find the uniformly most powerful test of exact significance level } \alpha = 0.05. \]

2. Let $X_1, \ldots, X_n$ be a random sample of size $n$ from the p.d.f.
   \[ f_\theta(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} 1_{(0,\infty)}(x), \quad x \in \mathbb{R}, \]
   for some $\theta > 0$, and let $\bar{X}$ denote the sample mean.
   \[ \text{a.} \quad \text{Compute } E_\theta(\bar{X}^2). \]
   \[ \text{b.} \quad \text{Find the uniform minimum variance unbiased estimator of } \varphi(\theta) = \text{Var}_\theta(X_1). \]
   \[ \text{c.} \quad \text{Check for } n = 1 \text{ whether the estimator in part b achieves the Cramér-Rao lower bound.} \]

3. Let $X_1, \ldots, X_8$ be independent and identically distributed random variables, and suppose that each $X_i$ has a standard normal distribution. Define $\bar{X}_1 = \frac{1}{4} \sum_{i=1}^{4} X_i$ and $\bar{X}_2 = \frac{1}{4} \sum_{i=5}^{8} X_i$.
   \[ \text{a.} \quad \text{What is the distribution of } \frac{1}{2}(\bar{X}_1 + \bar{X}_2) ? \]
   \[ \text{b.} \quad \text{What is the distribution of } 4\bar{X}_1^2 ? \]
   \[ \text{c.} \quad \text{What is the distribution of } \bar{X}_1^2/\bar{X}_2^2 ? \]
   \[ \text{d.} \quad \text{For a certain number } c > 0 \text{ the random variable } \bar{X}_1/\sqrt{c\bar{X}_2^2} \text{ has a student-type distribution. What is } c \text{ and what is the number of degrees of freedom?} \]

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4. Given is a random sample \(X_1, X_2, X_3\) of size 3 from a Normal \((\theta, 1)\) distribution, \(\theta \in \mathbb{R}\).
   a. Find a sufficient and complete statistic for \(\theta \in \mathbb{R}\).
   b. Find the uniform minimum variance unbiased estimator of \(\varphi(\theta) = \theta^2\). Explain!
   c. Determine \(E(X_1^2 - \frac{1}{2}(X_2 - X_3)^2|X)\), where \(X\) is the sample mean. Explain!
5. Suppose that \(X_1, \ldots, X_n\) are independent random variables and that \(X_i\) has density
   \[
   f_\theta(x) = \frac{1}{\theta t_i} e^{-\frac{x}{\theta t_i}} 1_{(0, \infty)}(x), \quad x \in \mathbb{R},
   \]
   where \(\theta > 0\) is an unknown parameter and \(t_i \neq 0\) a given number \((i = 1, \ldots, n)\).
   a. Determine the maximum likelihood estimator \(\hat{\theta}\) of \(\theta\).
   b. Determine the family of likelihood ratio tests for testing the null hypothesis \(H_0 : \theta = 1\) against the alternative \(H_1 : \theta \neq 1\), and prove that these tests are equivalent with rejecting \(H_0\) when either \(\hat{\theta} \leq c_1\) or \(\hat{\theta} \geq c_2\) for suitable numbers \(c_1 \leq c_2\) (\(\hat{\theta}\) as in a).
6. Consider the function
   \[
   f(x, y) = cx^2 y 1_D(x, y), \quad (x, y) \in \mathbb{R}^2,
   \]
   where \(D = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 < y < 1\}\).
   a. Determine the number \(c\), such that \(f\) is a p.d.f.
   Henceforth let \(c\) be the number obtained in a and suppose that \(X, Y\) are random variables with joint p.d.f. \(f\).
   b. Compute the marignal p.d.f. \(f_X\) of \(X\).
   c. Compute the conditional expectation \(E(Y|X)\) of \(Y\) given \(X\).
   d. Which of \(E(Y|X)\) and \(Y\) has the smaller variance?
7. Let \(X_1, \ldots, X_{100}\) be independent and identically distributed random variables, and let each \(X_i\) have p.d.f.
   \[
   f(x) = 2x 1_{(0,1)}(x), \quad x \in \mathbb{R}.
   \]
   Give an approximation for the probability that at least 20 of these random variables exceed \(2/\sqrt{5}\).
8. Let \(X\) and \(Y\) be independent and identically distributed, each with p.d.f.
   \[
   f_\theta(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} 1_{(0, \infty)}(x), \quad x \in \mathbb{R},
   \]
   where \(\theta > 0\) is an unknown parameter.
   a. Argue that \(X/(X + Y)\) is ancillary for \(\theta > 0\) (i.e. has a distribution that does not depend on \(\theta\)).
   b. Argue that \(X + Y\) and \(X/(X + Y)\) are stochastically independent for each \(\theta > 0\).