Work all 8 problems. Begin each problem on a new page, using one side of the sheet. Throughout “p.d.f.” means “probability density function”, $\mathbb{R} = (-\infty, \infty)$, and $1_A$ denotes the indicator of the set $A$ (i.e. $1_A(x) = 1$ if $x \in A, 0$ otherwise). A table of the standard normal distribution is attached.

1. Let $X_1, \ldots, X_n$ be independent random variables and assume that $X_i$ has a normal distribution with mean $c_i\mu, \mu \in \mathbb{R}$, and variance $\sigma^2 > 0$, where $c_i$ is a known number ($i = 1, \ldots, n$). Assume that $c_i \neq 0$ for at least one index $i$. The parameters $\mu$ and $\sigma^2$ are unknown.
   a. Find the maximum likelihood estimators of $\mu$ and $\sigma^2$.
   b. Find the family of likelihood ratio tests for testing the null hypothesis $H_0: \mu = 0$ against the alternative $H_1: \mu \neq 0$.
   c. For $c_1 = \cdots = c_n = 1$, show that the LR test statistic can be reduced to a statistic with a student distribution.

2. Suppose that $p$ and $q$ are p.d.f.’s that are strictly positive and continuous on the interval $[0, 1]$. Let $X_1, \ldots, X_n$ be a random sample of size $n$ from the p.d.f.
   \[
f_\theta(x) = c(\theta)\{p(x)\}^\theta\{q(x)\}^{1-\theta}1_{[0,1]}(x), \quad x \in \mathbb{R},
\]
   where the parameter $\theta \in (0, 1)$ is unknown and $c(\theta) = [\int_0^1 \{p(x)\}^\theta\{q(x)\}^{1-\theta}dx]^{-1}$.
   a. Show that the family of p.d.f.’s $f_\theta, 0 < \theta < 1$, is an exponential family.
   b. Determine the family of uniformly most powerful critical regions for testing the null hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: 0 < \theta < \theta_0$, for some given $\theta_0 \in (0, 1)$.

   Henceforth let $p(x) = 2x1_{[0,1]}(x), q(x) = 1_{[0,1]}(x), x \in \mathbb{R}, n = 2, \text{ and } \theta_0 = \frac{1}{2}$.
   c. Determine the density $f_{\frac{1}{2}}(x), x \in \mathbb{R}$, explicitly.
   d. Find, in the present situation, the most powerful test of size $\alpha, 0 < \alpha < 1$.

3. Let $X$ be a random variable with a double exponential p.d.f.
   \[
f_\theta(x) = \frac{1}{2\theta} e^{-|x|/\theta}, \quad x \in \mathbb{R},
\]
   for some unknown $\theta \in (0, \infty)$. Suppose $X_1, \ldots, X_n$ is a random sample of size $n$ from this p.d.f.
   continued on page 2
a. Compute $E_\theta |X|$ and $E_\theta X^2$.

b. Show that the statistic $T = \frac{1}{n} \sum_{i=1}^{n} |X_i|$ is an unbiased estimator of $\theta$, and calculate its variance.

c. Calculate the Fisher information $I(\theta)$ at $\theta$ for the family of p.d.f.’s given above.

d. Is $T$ in part b the uniform minimum variance unbiased estimator of $\theta$? Why?

4. Suppose that $X$ and $Y$ are jointly continuous random variables with 

$$f_{Y|X}(y|x) = 1_{(x,x+1)}(y), \quad f_X(x) = 1_{(0,1)}(x), \quad x, y \in \mathbb{R}. $$

a. Compute Cov($X, Y$).

b. Compute $P(X + Y < 1)$.

c. Find $f_{X|Y}(x|y)$ for $0 < y < 2$.

d. Find the conditional expectation $E(X|Y)$.

5. Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a given function that is strictly positive and continuous on $[0, \infty)$. Let $P(x) = \int_0^x p(y)\,dy, \quad x \in \mathbb{R}$.

a. For $\theta > 0$ determine the number $c(\theta)$ in such a way that the function 

$$f_\theta(x) = c(\theta)p(x)1_{[0,\theta]}(x), \quad x \in \mathbb{R},$$

is a p.d.f.

A random sample $X_1, \ldots, X_n$ of size $n$ from the density $f_\theta$ is given, where the parameter $\theta > 0$ is unknown.

b. Compute the maximum likelihood estimator of $\theta$.

c. Find a complete sufficient statistic for $\theta$. Explain why it is complete.

d. Find the minimum variance unbiased estimator of $P(\theta)$.

e. Determine the conditional expectation $E(P(X_1)|S)$, if $S$ is a complete sufficient statistic for $\theta$.

6. Suppose that the random variables $X_1, \ldots, X_n$ are independent and identically distributed with p.d.f.

$$f(x) = 2x1_{[0,1]}(x), \quad x \in \mathbb{R},$$

under the null hypothesis $H_0$, and p.d.f.

$$g(x) = 4x^31_{[0,1]}(x), \quad x \in \mathbb{R},$$

under the alternative $H_1$. 

continued on page 3
a. Determine the family of most powerful tests for testing $H_0$ against $H_1$.

b. Determine the most powerful test of approximate size $\alpha$, $0 < \alpha < 1$, assuming that the sample size $n$ is large enough for application of the central limit theorem.

7. Let the discrete random variables $X$ and $Y$ have the joint p.d.f.

$$f_{X,Y}(x,y) = \frac{2}{n(n+1)}, y = 1, \ldots, x, \ x = 1, \ldots, n.$$  

Compute:

a. The marginal p.d.f $f_Y$.

b. The conditional p.d.f. $f_{X|Y}(x|y)$.

c. The conditional expectation $E(X|Y)$.

d. Of course we have either $\text{Var}(X) \leq \text{Var}(E(X|Y))$ or $\text{Var}(X) \geq \text{Var}(E(X|Y))$.

Which of these two inequalities should hold according to theory?

8. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with common density

$$f(x) = 3 \ x^2 1_{[0,1]}(x), \ x \in \mathbb{R},$$

and let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Find the limiting normal distribution of $\sin(\bar{X}_n)$, as $n \to \infty$.

(Hint: you may use the “delta-method”.)