2015 May ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let \( A \) be an \( n \times n \) time-independent matrix.
   
   a) Prove that \( e^{At} \) is a fundamental matrix for the system \( \dot{x} = Ax \).
   
   b) Prove for any continuous function \( f(t) \), \( x(t) = e^{At}(C + \int_0^t e^{-A\tau} f(\tau) \, d\tau) \) is the general solution for the system \( \dot{x} = Ax + f(t) \).

   c) Let \( A = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 & -1 \end{bmatrix} \). Find \( e^{At} \).

2. Let \( f(t, x) \in C^1(\mathbb{R}^{n+1}) \). Prove that for all \( t \geq 0 \), if there is a function \( k(t) \) such that \( \|f(t, x)\| \leq k(t)\|x\| \) for all \( x \in \mathbb{R}^n \), then the solution \( x(t) \) of the initial value problem \( \dot{x} = f(t, x), \ x(0) = x_0 \) satisfies
   \[
   \|x(t)\| \leq \|x_0\|e^{\int_0^t k(s)ds}.
   \]

3. Investigate the stability of the origin of the system
   
   \[
   \begin{align*}
   \dot{x}_1 & = -x_2 + x_1x_3 \\
   \dot{x}_2 & = x_1 + x_2x_3 \\
   \dot{x}_3 & = -x_3 - x_1^2 - x_2^2 + x_3^2
   \end{align*}
   \]

4. Prove that the system
   
   \[
   \begin{align*}
   \dot{x}_1 & = x_2 \\
   \dot{x}_2 & = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)
   \end{align*}
   \]

   has a periodic orbit.
Do three out of four problems below. Clearly indicate in the following boxes which three problems to be graded; otherwise problems 1, 2, and 3 will be used for grading:

1. Let $U$ be a semi-infinite strip in $\mathbb{R}^2$ defined by
   
   $U = \{(x, y) \in \mathbb{R}^2 : 0 < y < L, 0 < x < \infty\}$.

   Let $u(x, y) \in C^2(U) \cap C(\bar{U})$ be a classical solution of the following problem
   
   \[ u_{xx} + u_{yy} = 0 \text{ in } U, \]
   
   \[ u(x, 0) = u(x, L) = 0 \text{ for all } x \in (0, \infty), \]
   
   \[ u(0, y) = 0 \text{ for all } y \in [0, L]. \]

   Assume that there are positive numbers $N$ and $C$ such that
   
   \[ |u(x, y)| \leq C + x^N \text{ for all } (x, y) \in U. \]

   Prove that $u(x, y) = 0$ for all $(x, y) \in U$.

2. Let $D = U \times (0, T]$, where $U$ is a bounded domain in $\mathbb{R}^n$ and $T > 0$.

   (a) State without proof the maximum principle for the classical solution of the heat equation $v_t - \Delta v = 0$ in $D$.

   (b) Prove that there is at most one classical solution $u \in C^{2,1}_{x,t}(D) \cap C(\bar{D})$ of the following initial boundary value problem:
   
   \[ u_t - \Delta u = |\nabla u|^2 \text{ in } D, \]
   
   \[ u(x, 0) = h(x) \text{ on } U, \]
   
   \[ u(x, t) = g(x, t) \text{ on } \partial U \times (0, T], \]

   where $h(x)$ and $g(x, t)$ are given initial and boundary data.

   (Hint: You can try to use $v = \Phi(u)$ with an appropriate function $\Phi$.)
3. Assume $u(x,t)$ is the unique classical solution of the following Cauchy problem

$$u_t = -xu_x \quad \text{in} \quad \mathbb{R} \times (0, \infty),$$

$$u(x,0) = \phi(x) \quad \text{for all} \quad x \in \mathbb{R},$$

where $\phi(x) \in C^1(\mathbb{R})$ is a given function that satisfies $\phi(0) = 0$.

Prove for any $x \in \mathbb{R}$ that

$$\lim_{t \to \infty} u(x,t) = 0.$$

4. Let $U$ be a bounded domain in $\mathbb{R}^n$. Suppose $u(x,t) \in C^2(\bar{U} \times [0, \infty))$ is a classical solution of the problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - \frac{\partial u}{\partial t} \quad \text{in} \quad U \times (0, \infty),$$

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0 \quad \text{on} \quad U,$$

$$u(x,t) = h(x,t) \quad \text{on} \quad \partial U \times [0, \infty),$$

where $h(x,t)$ is a given continuous function on $\partial U \times [0, \infty)$.

Assume there is a function $H(x,t) \in C^2(\bar{U} \times [0, \infty))$ such that $H(x,t) = h(x,t)$ on $\partial U \times [0, \infty)$, and $H(x,0) = H_t(x,0) = 0$ on $U$.

Prove that there exists a positive constant $C$ such that for any $t > 0$ one has

$$\int_0^t \int_U |u_t(x,\tau)|^2 \, dx \, d\tau \leq C \left( \int_0^t \int_U |H_t(x,\tau)|^2 \, dx \, d\tau + \int_0^t e^{-(t-\tau)} G(\tau) \, d\tau \right),$$

where

$$G(t) = \int_0^t \int_U F^2(x,\tau) \, dx \, d\tau$$

with $F(x,t) = \frac{\partial^2 H}{\partial t^2} - \Delta H + \frac{\partial H}{\partial t}$. 
