2015 August ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Find a fundamental matrix for the time-variant system

\[
\begin{align*}
\dot{x}_1 &= -x_1 - x_3 \\
\dot{x}_2 &= -2x_2 + 2tx_3 \\
\dot{x}_3 &= x_3
\end{align*}
\]

2. Let \( E \subset \mathbb{R}^n \) be open and \( f : E \rightarrow \mathbb{R}^n \) be continuously differentiable on \( E \). Prove that \( f \) is locally Lipschitz on \( E \).

3. Investigate the stability of the origin of the system

\[
\begin{align*}
\dot{x}_1 &= -2x_1 - 3x_2 + 2x_3^2 + x_1^2 - 2x_1x_3 \\
\dot{x}_2 &= x_1 + x_2 \\
\dot{x}_3 &= x_3^2 + x_1^2 - 2x_1x_3
\end{align*}
\]

by using the center manifold theorem.

4. Show that the following systems have no periodic orbit

a)

\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= 2x_1 + x_2 + (x_1 + x_2)(x_1 - x_2)
\end{align*}
\]

b)

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2^2, \\
\dot{x}_2 &= x_2(1 - x_1^2)
\end{align*}
\]
1. Let $D$ be a bounded domain in $\mathbb{R}^n$ with $C^1$-boundary, and $U = D \times (0, \infty)$. Let $\Gamma$ denote the parabolic boundary of $U$.

Let $\alpha_1, \alpha_2$ be two positive numbers.

Let $u_1(x, t)$ and $u_2(x, t)$ with $(x, t) \in \bar{U}$ be two functions in $C^2(\bar{U})$.

Suppose

$$\frac{\partial u_1}{\partial t} - \alpha_1 \Delta u_1 = \frac{\partial u_2}{\partial t} - \alpha_2 \Delta u_2$$
on $U,$

and

$$u_1 = u_2$$
on $\Gamma.$

Assume there is $M > 0$ such that

$$\int_D \left( |\nabla u_1(x, t)|^2 + |\nabla u_2(x, t)|^2 \right) dx \leq M \quad \forall t > 0.$$

Prove that there exists a constant $C > 0$ such that

$$\int_D |u_1(x, t) - u_2(x, t)|^2 dx \leq C |\alpha_1 - \alpha_2|^2 \quad \forall t \geq 0.$$

(Note: Poincaré’s inequality can be used without proof.)

2. Let $D$ be a bounded domain in $\mathbb{R}^n$ and $U = D \times (0, \infty)$. Let $\Gamma$ denote the parabolic boundary of $U$. Suppose $u(x, t)$ is a classical solution of the problem

$$u_t(x, t) - \Delta u(x, t) = 1 \quad \forall (x, t) \in U,$$

and

$$u(x, t) = 1 \quad \forall (x, t) \in \Gamma.$$

Prove that for any $\alpha > 1$ and $x \in D$ one has

$$\lim_{t \to \infty} \left( \frac{u(x, t)}{t^\alpha} \right) = 0.$$
3. Let $u(x, t)$ be the classical solution of the wave equation

$$u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty)$$

satisfying the initial conditions

$$u(x, 0) = g(x) \quad \text{and} \quad u_t(x, 0) = h(x),$$

where $g(x)$ and $h(x)$ are given functions on $\mathbb{R}$.

Assume

$$\lim_{x \to \pm \infty} g(x) = 0,$$

$$h(x) \geq 0 \quad \forall x \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} h(x) \, dx < \infty.$$

Prove for any fixed $t > 0$ that

$$\lim_{x \to \pm \infty} u(x, t) = 0.$$

4. Suppose $u(x)$ is a classical, bounded solution of the Laplace equation in $\mathbb{R}^n$, that is,

$$u \in C^2(\mathbb{R}^n), \quad |u(x)| \leq C \quad \text{and} \quad \Delta u(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^n,$$

where $C$ is a positive constant.

Prove that $u = \text{constant}$ on $\mathbb{R}^n$. 