FALL 2005 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. Given \( A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \)
   a) Discuss stability for the system \( \frac{dx}{dt} = Ax \).
   b) Find a fundamental matrix for the system in a).
   c) Solve the problem \( \frac{dx}{dt} = Ax + b(t) \) with \( x(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \) and \( b(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1 \end{bmatrix} \).

2. For the system of equations
   \[
   \begin{cases}
   x' = 2x - xy \\
   y' = -y + xy
   \end{cases}
   \]
   a) Find all equilibria.
   b) Prove that if \( x(0) > 0 \) and \( y(0) > 0 \) then \( x(t) > 0 \) and \( y(t) > 0 \) for all \( t \geq 0 \).
   c) Show that all solutions with \( x(0) > 0 \) and \( y(0) > 0 \) are bounded and, in fact, periodic when \( (x(0), y(0)) \neq (1, 2) \) (Hint: Use \( E(x, y) = x - \ln(x) + y - 2 \ln(y) \)).
   d) Determine stability properties of the equilibria in part a).

3. Suppose that \( \Phi(t) \) is a fundamental matrix for a linear system
   \[
   \frac{dx}{dt} = Ax.
   \]
   Suppose there exists \( M > 0 \) such that \( |\Phi(t)| \leq M \) for all \( t \geq 0 \). Show that \( x = 0 \) is stable.

4. Determine whether each of the following has a limit cycle or not:
   a) \[
   \begin{cases}
   x' = x^2 + 2y^2 \\
   y' = x - 2
   \end{cases}
   \]
   b) \[
   \begin{cases}
   x' = -12xy + x^3 \\
   y' = 4y
   \end{cases}
   \]
1. Consider the first order partial differential equation $u_t - e^{-u}u_x = 0$ (*)

   a) Solve the characteristic ODEs for (*).

   b) Find an explicit nonconstant solution of (*) in the form $u(x,t) = f(x/t)$. (Note: The solution you find may only be defined on a subdomain in the $x$, $t$ plane.)

2. a) Find the Green’s function for the boundary value problem (BVP)

   $$ y'' = 0, \ y(0) = 0, \ y(1) - 2y'(1) = 0. $$

   b) Use your answer in part a) to solve the BVP $y'' = x, \ y(0) = 0, \ y(1) - 2y'(1) = 0$.

3. Use Duhamel’s principle to solve the initial value problem for the non-homogeneous wave equation

   $$ w_{tt} = w_{xx} + \sin(x - t), \ x \in \mathbb{R}, \ t > 0, $$

   $$ w(x,0) = 0, \ x \in \mathbb{R}, $$

   $$ w_t(x,0) = 0, \ x \in \mathbb{R}. $$

4. Consider the following modified heat equation

   $$ u_t(x,t) = u_{xx}(x,t) - u(x,t), \ 0 < x < 1, \ t > 0, $$

   $$ u(x,0) = f(x), \ 0 < x < 1, $$

   $$ u(0,t) = 1, \ u(1,t) = 0, \ 0 < t < T. \quad (1) $$

   a) Find the steady state solution $u(x,t) = u_{ss}(x)$.

   b) Use an energy argument on the function

   $$ w(x,t) = u(x,t) - u_{ss}(x) $$

   to describe the behavior of $u$ as $t \to \infty$. 