PART I: ODE

1. Consider the Lorenz system
\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= xy - \beta z
\end{align*}
\] , with \( \rho > 0, \beta > 0 \) and \( \sigma \in (0, 1) \).

Show that the origin is globally asymptotically stable.

(hint: Use the Lyapunov function \( V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2 \).)

2. Consider a linear system \( \dot{x}(t) = A(t)x(t) \), \( x(t) \in \mathbb{R}^5 \) and fundamental matrix \( \Phi(t) \) with
\[
\Phi(t) = \begin{bmatrix}
\cos(t) & -\sin(t) & 0 & 0 & 0 \\
\sin(t) & \cos(t) & 0 & 0 & 0 \\
0 & 0 & \frac{4e^t - e^{-2t}}{3} & \frac{-2(e^t - e^{-2t})}{3} & 0 \\
0 & 0 & \frac{2(e^t - e^{-2t})}{3} & \frac{(4e^{2t} - e^t)}{3} & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

(a) Compute \( A(t) \). (hint: \( A(t) \) is actually a constant matrix.)

(b) Compute all periodic orbits and equilibria.

(c) Determine the stability of the system.

3. Consider the ordinary differential equation \( \dot{x}(t) = F(x(t), t) \), where \( x(t) \in \mathbb{R}^n \), and \( F : \mathbb{R}^n \to \mathbb{R}^n \) satisfies
\[
\|F(x, t)\| \leq L\|x\| + t, \quad \text{for all } x \in \mathbb{R}^n, \ t \in \mathbb{R}
\]
where \( L \) is a positive constant.

(a) Prove the following Gronwall’s Inequality: Let \( f_1(t), f_2(t), p(t) \) be continuous on \([a, b]\) and \( p \geq 0 \). If \( f_1(t) \leq f_2(t) + \int_a^t p(s)f_1(s) \, ds, \ t \in [a, b] \), then
\[
f_1(t) \leq f_2(t) + \int_a^t p(s)f_2(s) \exp\left(\int_s^t p(u) \, du\right) \, ds.
\]

(b) Obtain an explicit bound for \( \|x(t)\| \) in terms of \( L \) and \( t \) for all \( t \geq 0 \).

4. (a) For functions \( u \) and \( v \) in \( C^4[0, 1] \) that satisfy
\[
\begin{align*}
u(0) &= v(0) = 0, \ u'(0) = v'(0) = 0, \ u(1) = v(1) = 0, \ u''(1) = v''(1) = 0,
\end{align*}
\]
show that \( \int_0^1 \left( u \frac{d^4 v}{dx^4} - v \frac{d^4 u}{dx^4} \right) \, dx = 0 \).
(b) Show that the eigenfunctions corresponding to different eigenvalues of the problem
\[ \frac{d^4 \varphi(x)}{dx^4} + \lambda e^x \varphi(x) = 0, \quad x \in [0,1], \quad \varphi(0) = \varphi'(0) = \varphi(1) = \varphi''(1) = 0 \]
are orthogonal with respect to an appropriate weight function. What is the weight function?

PART II: PDE

1. Transform the equation
\[ x^2 u_{xx} - y^2 u_{yy} + 3xu_x - yu_y = 0, \quad x > 0, \quad y > 0, \]
into a canonical form.

2. Consider the partial differential equation
\[ u_{tt} + \Delta_x u + \varphi(x)u_t = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad \text{where } \varphi(x) \geq 0 \text{ for all } x \in \mathbb{R}^n. \]
Suppose that \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution. Show that
\[ \int_{B_R(x_0)} u_t^2(x,T) + \|\nabla_x u(x,T)\|^2_2 \, dx \leq \int_{B_R(x_0)} u_t^2(x,0) + \|\nabla_x u(x,0)\|^2_2 \, dx, \]
for all \( x_0 \in \mathbb{R}^n \), all \( R > 0 \) and for all \( T \) satisfying \( 0 < T < R \).

3. Given that \( u(x) \) is harmonic on
\[ \left\{ x = (x_1, x_2) : \frac{1}{2} \leq \|x\| \leq 1 \right\} \subset \mathbb{R}^2, \]
let
\[ M_1 = \max\{u(x) : \|x\| = 1\} \quad \text{and} \quad M_2 = \max\{u(x) : \|x\| = \frac{1}{2}\}. \]
Assume that \( M_2 < M_1 \) and \( u(1,0) = M_1 \). Show that
\[ \frac{\partial}{\partial x_1} u(x) \bigg|_{x=(1,0)} \geq \frac{(M_1 - M_2)}{\ln(2)}. \]
(Hint: Consider \( v_\epsilon(x) = u(x) - \epsilon \ln(\|x\|) \) and use the maximum principle.)

4. Solve
\[ u_{tt}(x,t) - u_{xx}(x,t) = t \sin(\pi x), \quad x \in [0,1], \quad t \geq 0, \]
\[ u(x,0) = \sin(3\pi x), \quad u_t(x,0) = \sin(5\pi x). \]