PART I: ODE

1. Let \( A_{n \times n} \) be a real matrix and let \( f : \mathbb{R} \to \mathbb{R}^n \) be a continuous function which is periodic with period \( T \). Consider the differential equation,
\[
\frac{dx(t)}{dt} = Ax(t) + f(t).
\]

a) Under the assumption that all eigenvalues of \( A \) have nonzero real part, show there exists a unique initial condition \( x_0 \) which produces a (unique) \( T \)-periodic solution \( x^*(t) \) to the equation (1).

b) If, in addition, all eigenvalues of \( A \) have strictly negative real parts, show that the periodic solution \( x^* \) is stable, i.e., if \( x \) is an arbitrary solution of (1) then \( (x(t) - x^*(t)) \xrightarrow{t \to \infty} 0 \).

2. For each of the following systems determine whether the origin is stable, asymptotically stable, unstable or totally unstable. State any theorems referred to in each case.

a) \[
\begin{align*}
\dot{x}_1 &= x_2 - x_1^5 \\
\dot{x}_2 &= -x_1 - x_2^3 - x_2^7
\end{align*}
\]

b) \[
\begin{align*}
\dot{x}_1 &= x_2 + x_1^5 \\
\dot{x}_2 &= -x_1 - x_2^3 + x_2^7
\end{align*}
\]

3. Let \( y : \mathbb{R} \to \mathbb{R} \) be a nonzero solution of the differential equation,
\[
8 \frac{d^2y(x)}{dx^2} - 8 \frac{dy(x)}{dx} + e^x y(x) = 0.
\]

a) Show that \( y \) has infinitely many zeros in the interval \((0, \infty)\) and at most one zero in the interval \((-\infty, 0)\).

b) Label the positive zeroes as \( x_1 < x_2 < \cdots < x_n < \cdots \). Show that \( (x_{n+1} - x_n) \xrightarrow{n \to \infty} 0 \).

4. Prove the following:

Theorem: Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a globally Lipschitz continuous function. Then, for each \( x_0 \in \mathbb{R} \) there exists a unique \( C^1 \) function \( x : \mathbb{R} \to \mathbb{R} \) (i.e., defined for all \( t \in \mathbb{R} \)) such that \( x(0) = x_0 \) and
\[
\frac{dx(t)}{dt} = f(x(t)), \quad \text{for all } t \in \mathbb{R}.
\]

5. Consider the linear time varying system,
\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + t & \sin(t) & e^{-t} \cos(t) \\ -e^t & -t + 2 & -e^t \\ e^{t \sin(t)} & e^{t \sin(2t)} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Let \( \Phi(t) \) denote a fundamental matrix of solutions.

a) Derive from first principles a differential equation satisfied by \( \det(\Phi(t)) \).

b) Use (a) to show that the system is unstable. You must carefully state the definition of type of stability applicable here.
1. Solve the partial differential equation
   \[ uu_x + u_t = x; \quad x \in \mathbb{R}, \]
   subject to \( u(x,0) = 0 \).

2. Solve the Initial Value Problem in \( \mathbb{R}^3 \) for the wave equation
   \[ u_{tt} = \Delta u, \quad x \in \mathbb{R}^3, \quad t \geq 0 \]
   subject to initial conditions \( u(x,0) = 0 \) and \( u_t(x,0) = x_3^3 \).

3. Consider the eigenvalue problem
   \[ \frac{d^2\varphi(x)}{dx^2} + 3x^2 \frac{d\varphi(x)}{dx} + \lambda \varphi(x) = 0 \]
   with boundary conditions \( \varphi(1) = 0 \) and \( \varphi(2) = 0 \).
   
   a) Carefully argue that there are infinitely many eigenvalues \( \{\lambda_n\}_{n=1}^\infty \) and corresponding eigenfunctions \( \{\varphi_n\}_{n=1}^\infty \) of the problem. State an orthogonality relationship satisfied by the eigenfunctions.
   
   b) Use the results from part (a) to derive a formal solution in the form of a “generalized” Fourier series involving \( \varphi_n \) for the partial differential equation
      \[ u_{tt} = u_{xx} + 3x^2u_x, \quad x \in [1,2], \quad t > 0 \]
      subject to \( u(1,t) = u(2,t) = 0 \) for all \( t \geq 0 \), \( u(x,0) = 0, \quad u_t(x,0) = g(x) \) for all \( x \in [1,2] \).
      
      You must derive formulae that will enable computation of coefficients in the Fourier series. You may assume without proof that the \( \lambda_n \) are strictly positive.
   
   c) Write out the explicit solution of the problem in part (b) when \( g(x) = 3\varphi_2(x) + 4\varphi_6(x) \) where \( \varphi_n \) are the eigenfunctions given in part (a).

4. Let \( u : \mathbb{R}^2 \to \mathbb{R} \) be a harmonic function.
   
   a) State the Mean Value Theorem for harmonic functions.
   
   b) Use the Mean Value Theorem to show that
      \[ |u(x)|^2 \leq \frac{1}{\pi R^2} \iint_{||y-x|| \leq R} u^2(y) \, dy_1 \, dy_2 \]
      for all \( x \in \mathbb{R}^2 \) and all \( R > 0 \). Here \( || \cdot ||_2 \) denotes the Euclidian norm.
   
   b) If \( u \) satisfies \( \iint_{||y|| \leq R} u^2(y) \, dy_1 \, dy_2 \leq \sqrt{R} \) for all \( R > 0 \), prove that \( u(x) = 0 \) for all \( x \in \mathbb{R}^2 \).

5. Let \( z = z(x,t) \) denote a solution of the heat equation \( z_t(x,t) - z_{xx}(x,t) = 0 \) in the region \( Q = \{(x,t) : 0 < x < \ell, \ 0 < t \leq T\} \) (with \( \ell > 0 \) and \( T > 0 \)) which is continuous in the closed region \( \overline{Q} \). Prove that the maximum of \( z \) is achieved on the initial line \( S_0 = \{(x,0) : 0 < x < \ell\} \) or on the boundary lines \( S_0 = \{(0,t) : 0 < t \leq T\}, S_\ell = \{\ell,t) : 0 < t \leq T\} \).