Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

**PART I: ODE**

1. Work parts (a) and (b).
   
   (a) Let \( \Omega \subset D \) be a compact positively invariant set for \( \dot{x} = f(x) \). Let \( V : D \to \mathbb{R} \) be a continuously differentiable function with \( \dot{V} \leq 0 \) in \( \Omega \). Let \( M \) be the largest invariant set in \( E = \{ x \in \Omega : \dot{V}(x) = 0 \} \). For this setting, give a precise statement of LaSalle’s Invariance Theorem.

   (b) Consider the system \( \{ \dot{x} = y^3 - xy^2, \dot{y} = -x + x^2y \} \). Apply the LaSalle Invariance Theorem to show that the origin is asymptotically stable and give an estimate for the region of attraction. (Hint: Consider \( V(x, y) = \alpha x^2 + \beta y^4 \).)

2. For \( n \times n \) matrices \( A \) and \( B \), prove that \( e^{(A+B)t} = e^{At}e^{Bt} \) for all \( t \) if and only if \( AB = BA \).

3. Use the transformations, \( v(x) = (1 + x)^{1/2}u(x) \) and \( y = \ln(1 + x) \) to completely solve the eigenvalue problem,

   \[
   \frac{d}{dx} \left[ (1 + x)^2 \frac{du}{dx} \right] + \lambda u = 0, \quad 0 < x < 1,
   \]

   \[
   u(0) = 0, \\
   u(1) = 0.
   \]

4. Do both parts (a) and (b).
   
   (a) Let \( f : [-a, a] \to \mathbb{R} \), with \( a > 0 \), be a continuous function satisfying

   \[
   \begin{align*}
   f(x) &> 0, \quad 0 < x < a \\
   f(x) &= 0, \quad x = 0 \\
   f(x) &< 0, \quad -a < x < 0
   \end{align*}
   \]

   Show that

   \[
   H(x, y) = \frac{1}{2}y^2 + \int_0^x f(\xi) d\xi
   \]

   is positive definite on \( D = \{(x, y) : -a < x < a, \quad -\infty < x < \infty \} \).

   (b) For \( \ddot{x} + f(x) = 0 \), where \( f \) satisfies (\( *) \), show that \( (x, \dot{x}) = (0, 0) \) is a stable critical point in the phase space \( D \subset \mathbb{R}^2 \) (here \( y = \dot{x} \)).

5. Find the Green’s function for,

   \[
   y'' - \gamma^2 y = 0,
   \]

   \[
   y'(0) = 0, \\
   y(1) = 0,
   \]

   where \( \gamma \) is a positive constant.
PART II: PDE

1. Prove that the only bounded harmonic functions on $\mathbb{R}^n$ are constants.

2. Solve the initial value problem,

$$u_{tt} - u_{xx} = 1, \quad t > 0, \quad x \in \mathbb{R}$$

$$u(x, 0) = \begin{cases} 1, & -1 \leq x \leq 1, \\ 0, & |x| > 1, \end{cases}$$

$$u_t(x, 0) = 0.$$

3. Find the solution $u(x, y, z)$ of the equation,

$$xu_x + yu_y + u_z = u$$

with Cauchy data, $u(x, y, 0) = \phi(x, y)$ where $\phi$ is a given $C^1$ function.

4. Find the solution of the initial-boundary value problem,

$$u_t - u_{xx} - 2u_x = 0, \quad t > 0, \quad 0 < x < 1,$$

$$u(0, t) = u(1, t) = 0,$$

$$u(x, 0) = e^{-x} \sin(11\pi x).$$

5. Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^2$ boundary $\partial\Omega$. Let $u_0$ and $u_1$ be given $C^2$ functions on $\Omega$. Show the problem,

$$u_{tt}(x, t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}, \quad x = (x_1, x_2, \ldots, x_n) \in \Omega, \quad t > 0$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

$$u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

has at most one solution $u \in C^2(\overline{\Omega} \times [0, \infty))$. 