1. Let \( x \in \mathbb{R}^n \) with \( x = [x_1, x_2, x_3, \ldots, x_n]^T \) and \( x_1 \neq 0 \). Let \( u = x + \sigma e_1 \), where \( \sigma = \text{sign}(x_1)\|x\|_2 \), and let \( \theta = \frac{1}{2}\|u\|_2^2 \). Finally, let \( U = I - \frac{1}{\theta}uu^T \). Prove that \( U \) is unitary and that \( Ux = -\sigma e_1 \).

2. Let \( A \in \mathbb{R}^{n \times n} \).
   (a) Prove that the trace of \( A \) equals the sum of its eigenvalues.
   (b) Prove that if the eigenvalues of \( A \) satisfy \(|\lambda_1| > |\lambda_i|\) for \( i = 2, 3, \ldots, n \), then
   \[
   \lambda_1 = \lim_{m \to \infty} \frac{\text{tr}(A^{m+1})}{\text{tr}(A^m)}.
   \]

3. Let \( A \) be an \( n \times n \) matrix and let \( Q = L + D \) be the lower triangular part of \( A \), including the diagonal. Prove that if \( A \) is strictly diagonally dominant, the Gauss-Seidel method
   \[
   Qx^{(k+1)} = (Q - A)x^{(k)} + b, \quad k = 0, 1, 2, \ldots
   \]
   converges to the solution of \( Ax = b \) for any starting vector \( x^{(0)} \).

4. Consider the equation \( x^3 - x - 1 = 0 \) which has a root \( \xi \) between 1 and 2.
   (a) Determine a suitable iteration function \( T(x) \) such that \( \xi \) is a solution of \( x = T(x) \) and \( T(x) \) is a contraction over \([1, 2] \).
   (b) Find \( k \) such that the \( n^{th} \) iterate \( x_n \) generated by the equation \( x_n = T(x_{n-1}) \) for \( n \geq 1 \), satisfies \( |x_n - \xi| \leq k^n|x_0 - \xi| \).

5. Let \( f(x) \) be the circular quarter arc given by \( f(x) = \sqrt{1 - x^2}, 0 \leq x \leq 1 \). Approximate \( f(x) \) by a straight line \( p_1(x) \) in the least squares sense using the weight function \( \rho(x) = (1 - x^2)^{-1/2}, 0 \leq x \leq 1 \). That is, using the inner product
   \[
   (u, v) = \int_0^1 u(x)v(x) \sqrt{1 - x^2} \, dx.
   \]

6. Let \( f \in C[1, 2] \) and let \( P^n \) be the set of polynomials of degree \( \leq n \). Define an inner product on \( C[1, 2] \) as \((f, g) = \int_1^2 x^2f(x)g(x) \, dx \) and norm \( \|f\| = (f, f)^{1/2} \). Let \( \phi_k(x) \) be orthonormal polynomials with respect to this inner product. The least squares approximation \( p_n \) to \( f \in C[1, 2] \) is given by \( p_n(x) = \sum_{k=0}^n (f, \phi_k)\phi_k(x) \). Prove that \( (f - p_n, q_n) = 0 \) for any \( q_n \in P^n \).
7. Let $Q(f)$ be the $(N + 1)$-point Gaussian quadrature rule over the interval $[a, b]$ such that

$$Q(f) = \sum_{i=0}^{N} w_i f(x_i) \approx I(f) = \int_{a}^{b} \rho(x)f(x)dx,$$

where $\rho(x)$ is a real, positive weight function on $(a, b)$. Show that if $a$ and $b$ are finite and $f$ is continuous, then $Q(f) \to I(f)$ as $N \to \infty$.

8. Consider the following two-step method,

$$y_{k+1} + \alpha_0 y_k = h (\beta_1 f(t_k, y_k) + \beta_0 f(t_{k-1}, y_{k-1})),$$

for solving the initial value problem $y'(t) = f(t, y)$.

(a) Find $\alpha_0, \beta_0, \beta_1$ such that the method is second order.

(b) Clearly define consistent, absolutely stable, region of absolute stability, and $A$-stable.

(c) Is the given two-step method consistent? Why or why not?

9. Consider the nonlinear boundary value problem,

$$-u'' = \cos(u),$$

posed on $(0, 1)$ with boundary conditions $u(0) = u(1) = 0$.

(a) Write down the nonlinear algebraic system of equations resulting from the finite difference method with $N$ internal nodes.

(b) Consider the iterative strategy of solving the nonlinear system from (a) with an initial solution vector $u^{(0)}$ and iterating $A u^{(n+1)} = F(u^{(n)})$, where $A$ is the finite difference matrix obtained by discretizing $-u''$ and $F(u^{(n)})_i = \cos u_i^{(n)}$. Show that this iteration converges to the solution of the algebraic equation for any initial input. (Hint: the fact that the eigenvalues of $A$ are known to be $\{2N^2 \left(1 - \cos \left(\frac{\pi j}{N+1}\right)\right)\}_{j=1}^{N}$ may be helpful in determining the norm of $A^{-1}$ and/or $A$.)