Do all nine problems.

1. Let $\Omega = [a, b]$ be a compact interval in $\mathbb{R}$, and let $P^N$ be the space of real polynomials of degree at most $N$. Suppose $p_N$ is the best approximation to $f \in C^{(N+1)}(\Omega)$ from $P^N$ in the $L^2$ norm defined by

$$\|v\|_2 = \sqrt{\int_a^b v(x)^2 \, dx}.$$

Find an upper bound on $\|p_N - f\|_2$.

2. The Hermite polynomials obey the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and are orthogonal with respect to the inner product

$$(u, v) = \int_{-\infty}^{\infty} e^{-x^2} \, u(x) \, v(x) \, dx.$$

(a) Find the nodes and weights for the two-point Gaussian quadrature rule for approximation of

$$I(f) = \int_{-\infty}^{\infty} e^{-x^2} f(x) \, dx.$$

You may need the formula $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$.

(b) For what space of functions is this two-point rule exact?

3. Let $A$ be an $n \times n$ real symmetric positive definite matrix. Prove that the minimum value of the Rayleigh quotient

$$R(x) = \frac{x^T A x}{x^T x}$$

over all $x \in \mathbb{R}^n \setminus 0$ is equal to the minimum eigenvalue of $A$. If you like, you may make the simplifying assumption that $A$ has distinct eigenvalues.

4. Prove that the equation

$$x = \cos(x)$$

has a unique solution, and that fixed point iteration

$$x_{n+1} = \cos(x_n)$$

converges to that solution starting from any initial guess $x_0 \in \mathbb{R}$. 


5. Find the Cholesky factorization of
\[
\begin{pmatrix}
1 & 1 & 3 \\
1 & 5 & 5 \\
3 & 5 & 11 \\
\end{pmatrix}.
\]

6. Let \(A\) be an \(m \times m\) real symmetric positive definite matrix, and let \(b\) be a vector in \(\mathbb{R}^m\). Let \(x_\ast\) be the solution to \(Ax_\ast = b\), and let the function \(f : \mathbb{R}^m \to \mathbb{R}\) be defined as
\[
f(x) = \frac{1}{2}x^T Ax - x^T b.
\]
Prove that \(f\) has a unique minimum at \(x_\ast\).

7. Let \(A\) be a matrix of size \(m \times n\). Prove that \(\|A\|_2\) is equal to the largest singular value of \(A\).

8. The midpoint method approximates the solution of the initial value problem
\[
y' = f(x, y) \quad y(x_0) = y_0
\]
using the step formula
\[
\begin{align*}
\tilde{y}_{n+\frac{1}{2}} &= y_n + \frac{h}{2}f(x_n, y_n) \\
x_{n+\frac{1}{2}} &= x_n + \frac{h}{2} \\
y_{n+1} &= y_n + hf\left(x_{n+\frac{1}{2}}, \tilde{y}_{n+\frac{1}{2}}\right) \\
x_{n+1} &= x_n + h.
\end{align*}
\]
(a) Find the local truncation error of this method. State whatever differentiability assumptions are needed.
(b) Determine whether this method is A-stable.

9. An \(m \times m\) matrix \(A\) is called strictly row diagonally dominant if, for each \(i = 1, 2, \ldots, m\), the elements in row \(i\) obey the inequality
\[
|A_{ii}| > \sum_{j \neq i} |A_{ij}|.
\]
(a) Prove that strict row diagonal dominance of \(A\) implies that \(A\) is nonsingular.
(b) Let \(D\) be the diagonal part of \(A\). Jacobi’s iteration for solving \(Ax = b\) is
\[
x^{(n+1)} = D^{-1}b - D^{-1}(A - D)x^{(n)}
\]
starting from an initial guess \(x^{(0)}\). Prove that if \(A\) is strictly row diagonally dominant, then Jacobi’s iteration converges (in every matrix norm) to the unique solution of \(Ax = b\).