1. Consider a nonsingular matrix $A \in \mathbb{C}^{n \times n}$.

(a) Write out the steps of the ("pure" unshifted) QR algorithm for finding the eigenvalues of $A$.

(b) Show that each of the matrices $A_k$ generated by the QR algorithm is unitarily similar to $A$.

(c) Show that if $A$ is upper Hessenberg, then so are each of the matrices $A_k$ generated by the QR algorithm.

(d) Describe an algorithm for computing an SVD of $A$ using the eigenvalue decomposition of a related matrix.

2. Let $A$ be the $3 \times 3$ matrix be

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 14 \end{bmatrix}$$

(a) Compute the Cholesky factorization of $A$.

(b) Let $U$ be an $n \times n$ upper triangular matrix with no zeros on the diagonal, and let $b$ be a vector in $\mathbb{R}^n$. Derive the exact number of floating point operations required to solve the triangular system of equations $Ux = b$ for $x$.

3. Estimate the numerical value of the integral

$$I = \int_{-1}^{1} \frac{dx}{1 + x^4}$$

with three-point Gauss-Legendre quadrature.

4. Let $p_N$ be the best $N$-th degree polynomial approximation to a real-valued function $f \in L^2[0, 1]$ in the norm

$$\|v\|_2 = \sqrt{\int_{0}^{1} v^2 \, dx}.$$

(a) Compute the best first-degree polynomial approximation to $f(x) = x^2$.

(b) Let $N = 1$, and suppose that $f$ is continuous. Prove that there is at least one point $\xi \in [0, 1]$ such that $p_1(\xi) = f(\xi)$.

(c) Suppose $f \in C^{N+1}[0, 1]$. Derive an estimate for the uniform norm

$$\|p_N(x) - f(x)\|_{\infty}.$$
5. Consider numerical solution of the initial value problem \( y' = f(t, y), \ y(0) = y_0 \) with stepsize \( h \).
A step of the \textit{theta method} is as follows:

\[
t_{n+1} = t_n + h
\]
\[
y_{n+1} = y_n + \theta hf(t_n, y_n) + (1 - \theta) hf(t_{n+1}, y_{n+1})
\]
where \( \theta \) is a constant parameter in the interval \([0, 1]\).

(a) After making the minimum necessary assumptions about the differentiability of \( f \), find the \textit{order of accuracy} for this method as a function of \( \theta \). State your differentiability assumptions clearly. Note that your answer will depend on \( \theta \).

(b) Clearly state the definitions of absolute stability and A-stability. For what (if any) values of \( \theta \) is this method A-stable?

6. Consider the fixed point iteration \( x_{k+1} = g(x_k) \), where

\[
g(x) = \tan^{-1}(2x).
\]

(a) Clearly \( g(x) \) has a fixed point at \( x = 0 \). Show that the fixed point iteration will not converge to this fixed point.

(b) There is another fixed point \( x_* \) near \( x = 1.166 \). Find an appropriate interval about \( x_* \) and prove that the fixed point iteration will converge to \( x_* \) for any initial guess \( x_0 \) in this interval. [Hint: \( \tan^{-1}(2) \approx 1.107 \).]

(c) Write out a step of Newton’s method for finding \( x_* \). (You do not need to show convergence for Newton’s method.)